

Issues on Problem Solving in Mathematics

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ABSTRACT

This paper is based on the issues about pedagogical and cognitive aspects of problem solving and explored ways to lessen the heavy cognitive load of a problem solving task. It established a problem type schema for students at different levels. It recognized the role of modern technology as a cognitive tool that promotes learning mathematics with understanding. It designed the framework of a techno-mathematics curriculum for algebra at the collegiate level.

INTRODUCTION

On the basis of modern civilization requires relentless quantification and critical evaluation of information in daily transactions, it becomes necessary to develop newer ways of thinking and reasoning that can be used to learn and do mathematical activities. Through problem solving for instance, we acquire a functional understanding of mathematics needed to cope with the demands of society. School mathematics of the twenty first century is viewed by educators to be that which should engage a learner in problem solving and reasoning. It should also foster deep understanding and develop the learner's critical and analytical thinking. Instruction should not be limited to plain mastery of algorithms or the development of certain mathematical skills. It should involve learners in investigation through "exploring, conjecturing, examining and testing" (NCTM, 1990, p.95). It should be tailored to promote reflective thinking among students. A wealth of research on mathematics education and cognitive science in the last decade has dealt with the pedagogical and cognitive aspects of

problem solving. Rivera and Nebres (1998) note specifically "the numerous published research studies of Fennema and Carpenter on Cognitively Guided Instruction (CGI) in the last quarter of this century [which] point to the pernicious effects of status quo. Developing critical and analytical thinking involves pedagogical conceptions with a philosophical basis. This paper adheres to the constructivist theory of learning and promotes the belief that problem solving processes rest on basic thinking skills which are best developed within a constructivist framework ways of thinking about mathematics and problem

solving (i.e. existing mathematics culture CGI recognizes the "acculturation of school children to an algorithmic approach to learning basic arithmetical facts" which pervade the current school mathematics culture and which have been proven to be "detrimental to children's own ways of thinking about problem solving and computations.

THE LEARNER AND COGNITIVE PROCESSE

Recent research studies on mathematics education have placed its focus on the learners and their processes of learning. They have posited theories on how learners build tools that enable them to deal with problem situations in mathematics. Blais reveals that the philosophical and theoretical view of knowledge and learning embodied in constructivism offers hope that educational processes will be discovered that enable students to acquire deep understanding rather

than superficial skills. (Blais, 1988, p.631) As learners experience their power to construct their own knowledge, they achieve the satisfaction that mathematical expertise brings. They acquire the ability to engage in critical and analytical context of reflective thinking. They develop systematic and accurate thought in any mathematical process. O'Daffer and Thorquist (1993) define critical thinking as "a process of effectively using skills to help one make, evaluate and apply decisions about what to believe or do"(p.40). They cited the observations of Facett(1938) on a student using critical thinking as one who

1. Selects the significant words and phrases in any statement that is important and asks that they be carefully defined;
 2. Requires evidence supporting conclusions she is pressed to accept;
 3. Analyzes that evidence and distinguishes fact from assumption;
 4. Recognizes stated and unstated assumptions essential to the conclusion;
 5. Evaluates these assumptions, accepting some and rejecting others;
 6. Evaluates the argument, accepting or rejecting the conclusion;
 7. Constantly reexamines the assumptions which are behind her beliefs and actions
- Experts seem to readily categorize the mathematical information in the material being read, thus facilitating

the processing of information that lead to the correct solution. They are able to attain some sort of a visual form of say an algebraic expression and are able to communicate this before they perform the algorithmic activity. Besides, they can determine errors and attain a deep understanding of the underlying structure of the mathematical concept. Experts rely not only on concepts and procedures when confronted with a mathematical problem. They also have access to metacognition which is the knowledge used by experts in "planning, monitoring, controlling, selecting and

evaluating cognitive activities" (Wong, 1989, Herrington, 1990, English, 1992 as cited by English-Halford, 1992; Bernardo, 1997). With this higher order thinking skill, problem solvers are assured of the success of every mathematical strategy they employ.

ISSUES ON TEACHING AND LEARNING PROBLEM SOLVING

Smith, Knudsvig and Walter (1998) advocate a cognitive schema which learners can use to acquire critical thinking strategies. They call it the TCDR for

TOPIC-CLASS-DESCRIPTION-RELEVANCE.

Thus, when given a learning material, students should ask the following questions:

- ☐ ☐What TOPIC I must understand?
- ☐ ☐What overall CLASS does this topic belong?
- ☐ ☐What is the DESCRIPTION of the topic?
- ☐ ☐What is the RELEVANCE of the topic?

These questions help learners interpret, analyze, organize and make sense of the information that are given in the material for better processing of learning. Once this becomes the framework of the learners, they gain strength and clarity of thinking. Several schemes have been offered by mathematics educators for solving word problems.

The most versatile and widely used scheme for problem solving is the one formulated by George Polya (1957). These include working simpler problems, restating a problem, decomposing or recombining a problem, drawing figures, making charts or organized lists, exploring related problems, using logical deduction, using successive approximations, using guess-and-check methods, and working backwards. (NCTM, 1989, as cited by Barb and Quinn, 1988, p. 537) Polya (1957) also developed a framework for problem solving in terms of such general phases as "understanding the problem, devising a plan,

carrying out the plan and looking back” (cited by Barb and Quinn, 1997, p. 537). If carried out effectively, then the problem solver becomes successful in handling a problem situation. But the process involved in traversing these steps is quite complex. The learner has to use her prior knowledge, apply acquired mathematical skills, understand the context of the problem situation, and choose the appropriate strategy in solving the problem. This requires formal abstraction, a higher order thinking skill that is available to experts alone. What, then, can be done to help novices gain intellectual power?

Another issue that is worth considering is the question of when students should engage in word problems. Word problems are usually treated as application problems since they are given after certain mathematical concepts are introduced, with the aim of using the concepts in solving the problems. On the other hand, word problems may be taught in context, i.e. they may be used to teach a mathematical idea or process. According to Laughbaum (1999) “[t]eaching in context also uses problems or situations, but they are used at the beginning of a math topic for the purpose of helping students understand the mathematics to be taught, or to create a motivating experience of the mathematics to follow” (p.1). Certain groups looked into the effects of application problems to the development of the skills of the learners. One such group called the Cognition and Technology Group of Vanderbilt (CTGV) identified the shortcomings of the application problems and came up with efficient ways of teaching word problems in context. The CTGV has these to say about application problems:

1. Instead of bringing real world standards to the work, students seem to treat word problems mechanically and often fail to think about constraints imposed by real-world experiences.
2. Single correct answers to application problems lead to misconceptions about the

nature of problem solving and inadvertently teaches students for a single answer rather than seek multiple answers.

3. The goal of one’s search for a solution is to retrieve previously presented information rather than rely on one’s own intuition. This may limit the development of people’s abilities to think for themselves.

4. They explicitly define the problems to be solved rather than help students to learn to generate and pose their own problems. Mathematical thinkers tend to generate their own problems.

5. The use of application problems lead to inert knowledge. Inert knowledge is that which is accessed only in a restricted set of contexts even though it is applicable to a wide variety of domain. (1997, p. 40)

There are other ways of facilitating recognition of problem structures, one of which is the use of text editing skills. In this activity, problem solvers are asked to identify missing information from problems or point out information that are irrelevant to the problems. Low and Over (1989) showed the significantly high correlation between students’ ability to edit the text of algebraic story problems and their ability to solve these problems; as well as between students’ ability to edit the text and categorize problems as being similar or different from each other (cited by Putt and Isaacs, 1992, p. 215). This activity enhances the problem solvers’ awareness of their own thinking processes. Such awareness helps learners identify their points of strengths and weaknesses and regulate their own ways of knowing.

In the light of all the issues and conflicts on various aspects of problem solving, particularly on developing cognitive strategies among students, and with the assumption that teachers hold wholesome beliefs and attitudes towards mathematics teaching, this paper attempts to offer suggestions on effective ways of fostering critical and analytical thinking

through problem solving at different school levels.

FOSTERING CRITICAL AND ANALYTICAL THINKING THROUGH PROBLEM SOLVING

At this point, we all agree that an expert problem solver is a critical and analytical thinker. When a learner gains expertise, she has acquired all the qualities of strong and smart thinking. She becomes insightful, and logical. The expert is also a constructive learner. She participates actively in the learning process and is able to build from her prior knowledge while assimilating and accommodating new knowledge. She appreciates the variety of ways of solving mathematical problems and recognizes a good solution. She is not afraid to use intuition and logic in her solutions. She makes good models of the problems and recognizes the essence and structure of a given problem. She employs a cognitive schema that helps her organize and plan her strategies. Her metacognitive skills help her monitor and evaluate her

progress. Expertise can be attained at an early age. Blais (1988) cites indicators of a schooler's expertise

once a teacher expresses doubt in her work. According to Blais, [I]f the child does not erase, if she or he refuses to accept the hint from an outside authority and tries to ponder whether the answer is correct, that student is an expert. Being willing and able to think and act independently, she or he will decide what is sensible and reasonable based on informal concepts already acquired (Mills, 1859). A child accustomed to accepting rules and procedures on faith has subordinated his or her own reasoning to outside authority and would have yielded to it once again; the child would have erased. (Blais, 1988, p. 626)

USE OF PROBLEM POSING

We have seen the benefits of acquiring problem type schemata in problem solving

activities. Recognition of the structures of the problem leads to the recognition of the essence of the problem. This promotes reflective abstraction and consequently critical thinking. Moses, Bjork and Goldenberg (1990) give the following suggestions on how the experiences of middle school students in problem solving can be enriched using problem posing:

1. Have students learn to focus their attention on known, unknown and restrictions of the problem. Then consider the following question: What if different things were known and unknown? What if the restrictions were changed.

2. Begin in comfortable mathematical territory.

3. Encourage students to use ambiguity to create new questions and problems.

4. Teach the idea of domain from the earliest grades, encouraging children to "play the same mathematical game with a different set of pieces. (Moses, Bjork and Goldenberg, 1990, pp. 83-86) Problem posing can also be applied to students using a variety of mathematical tasks that fit their interest and capacity. Various versions of problem posing and problem formulation activities are developed by mathematics educators, educational psychologists and cognitive scientists. An activity developed by Wilson, Fernandez and Hadaway (1993, p. 65) consists of making students list down the attributes of a given mathematical theorem or rule. Then the students are asked to generate new problems if some or all of the given attributes are not true..

The study of Bernardo (1998) used a kind of strategy in problem posing that promotes analogical transfer among high school students. They were given four types of basic probability problems. For each problem type, four analogous problems were developed. The students were given instructions on the solutions of the problems for each problem type. Students of the experimental group were asked to make

their own problems similar to the one they studied. Suggestions on objects and events

they can use in the problem were given. Then, the students were asked to solve the problem. The study showed that students who used the problem construction strategy were better at solving the analogous word problems. His study confirms an earlier research which he did in 1994 which showed that “problem solvers retain problemspecific information in problem-type representations because such information affords access to abstract structural information about the problems” (p. 392). His studies clarified the valuable contribution of problem-specific information in the process of acquiring abstract problem-type representation in the learner.

USE OF TECHNOLOGY IN MATHEMATICS INSTRUCTION

Mathematics classrooms in many places especially in progressive countries have access to computing technologies and other peripheral devices. Classroom equipment includes scientific and graphic calculators, calculator-based ranger, graph-link, calculator-based laboratory which includes motion detector, microphone, sensors and numerous other gadgets, computers, modems, printers, scanners, word processors, internet browser, electronic mail browser, CD-ROM and other interactive computer-based tools, televisions, video disc players, and all sorts of tools for recording and manipulating information. A lot of research has been conducted on the use of computer technology in education. There is a proliferation of calculator and/or computer based instructional materials in mathematics. Computer-based materials may come in the form of electronic information that can be retrieved from the World Wide Web or as a software designed computer-assisted instruction. There are sites which are devoted to the resources for teaching mathematics. Great attention has been given to computer assisted mathematics instruction, but “little is known about instructional design issues that affect students’ learning with technology”

(Wine & Stockley,1998, p107). One research project that developed high quality materials to support learning is the Jasper project which was conducted by the Cognition & Technology Group at Vanderbilt (CTGV) for 7 years. The Jasper series consists of 12 videodisc-based adventures with video-based analogs, extensions and teaching tips for use in mathematics instruction from the middle school to the higher levels. The eight features of the Jasper adventures are as follows:

1. Help students learn mathematics while solving problems in authentic context. The use of mathematics in authentic contexts supports students’ reasoning, problem solving, and communication skills, all standards identified by the NCTM (1989).
2. Provide a context that helps students integrate concepts in mathematics as well as mathematical knowledge with knowledge of other subjects.
3. Take advantage of the power of video and interactive technologies. Video allows a more veridical representation of events than text. It is dynamic, visual, and spatial, and students can more easily form rich mental models of the problem situations (e.g., Johnson-Laird, 1985; Mc.Namara, Miller & Bransford, 1991; Sharp et al., 1995).
4. Support Inquiry. The adventures are designed to help students understand the kinds of problems that can be solved through mathematical inquiry. The adventures also include embedded teaching that often takes the form of modeling by experts (Brown, Collins, & Duguid,1989). Modeling can also provide coaching and scaffolding for students as they develop their own skills (e.g., Vygotsky, 1978, 1986).
5. Students must generate as well as solve problems. The adventures end with challenges that specify a general goal for the students. Nevertheless, in order to solve the challenges, students must identify a number of subproblems and generate subgoals of their own.

6. Provide opportunities for collaboration over an extended period of time. As students work together over multiple class periods (from several days to several weeks) to solve a challenge, they have repeated opportunities to communicate about mathematics, share their ideas about problem solving, and receive feedback that helps them refine their thinking.

7. Afford students the opportunity to develop a deep understanding of mathematical concepts. Each videodisc adventure also includes video-based analog and extension problems. These problems help students engage in what-if thinking by revisiting the original adventures from new points of view.

8. Provide positive role models. A goal of the Jasper series is to provide positive role models for students from all backgrounds. (CGTV, 1997, pp.3-8)

Researchers attest to the success of the projects on technology-based mathematical instruction. Educators recognize the partnership that teachers and students can establish with computing technologies for effective mathematics teaching and learning. The best use of computing and multi-media technologies is in the context of support for mathematics instruction. This has to go with pedagogical principles that are deeply rooted in sound philosophies of knowledge and education. At this point, let us look into some college mathematics programs that used computing and multi-media technologies.

A TECHNOLOGY BASED CURRICULUM IN COLLEGE ALGEBRA

Problem solving is seen as the manipulation of an internal mental model of the external world. In the process of finding the solution, “we solve the problem in the internal representation and then project its solution into the thing being represented” (Hunt, 1994, p. 218). The solution is brought about by the manipulation of the representation by a human and/or an electronic thinking device. Learners construct a mental model of the situation in their

memory. The learner’s symbolic representation and manipulation is a limiting feature of human problem solving, though. Newell and Simon (1961,1972) proposed that computer programs can be gleaned as models of human thought and then offered the following insights:

1. A theory of the process of problem solving can be expressed as a program, that is a set of rules for manipulating symbols. Indeed, if a theory is proposed that cannot be so expressed, that theory is unacceptably vague.

2. The development of an ideal problem-solving program in some field of endeavor is a goal in Artificial Intelligence.

3. A problem- solving program that, in some nontrivial sense, behaves like a human being, is a descriptive theory of human problem solving. (cited by Hunt, p. 218) It becomes clear that technology is an efficient partner of humans in problem solving. Since the success in problem solving is determined by the learner’s capacity to represent an external situation into symbols and manipulate these representations, then they have to make use of some cognitive tools in the process. Cognitive scientists believe that learners usually memorize a variety of schemata in order to cope with the problem solving task. This is where the partnership between technology and humans becomes

essential. Hunt (1994) believes that “as long as the students have pattern-recognition rules that tells them when to apply which of their many contradictory schemata” then the problem solving skill has been acquired. They need not have an orderly progression of schemata like what computer programs have. More important than the procedures and algorithms is the meaningful understanding of the concepts applied in problem solving. That way, “[s]chemata problem solving works because it moves the computational burden from immediate memory, where the human problem solver is weak, to long-term memory, where the problem solver is strong” (Hunt,

1994, p.231). Since problem solving requires higher level cognitive skill, any mathematics course becomes meaningful if embedded with problem solving tasks.

CONCLUSION

Mathematics educators recognize the need to develop critical and analytical thinking through problem solving. This paper presented the various issues about problem solving that have been raised in the last two decades. Upon analyzing all arguments, this paper embraced the belief that establishing a cognitive schema in problem solving will lessen the heavy cognitive load of the problem solving task. Then this paper suggested ways to establish problem type schema among the students at different levels. In teaching problem solving at the elementary level, certain practices of the Cognitively Guided Instruction project may be employed. This includes awareness of problem schemata typology that teachers employ in class and knowledge of developmental solution strategies in assessing learner's solutions to problems. Another activity that enhances schema recognition is the problem posing task. The problem posing tasks are varied and have been proven to promote analogic transfer among the learners. The paper pointed out the importance of gaining ample knowledge in problem solving for critical thinking to take place in that particular setting. The paper showed how alternative solutions to problems can be encouraged using logic, reasoning, approximation, estimation and visual representations. These alternative solutions allow novices to harness their intuition to gain the expertise needed in problem solving. This way, they can take active part in building knowledge and gain expertise in the process. The role of technology as a cognitive tool and partner in mathematics instruction was recognized. Some research-based projects on the use of technology in mathematics instructions were cited. Some programs on problem-based

mathematics courses at the Ohio State University were also cited. These programs affirmed the benefits of the use of modern technology in promoting meaningful learning. A framework of a problem based curriculum for college algebra was recommended. While it is believed that computing and information technologies facilitate learning in a mathematics course, the use of modern technology is not vital in the proposed curriculum. Instead, it emphasizes the learning theories and pedagogical aspects of the curriculum, based on the constructivist theory of active building of knowledge. that promotes learning with understanding.

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