Sequence and Series

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Abstract

This research paper is a general overview of Sequence and Series. In this, we have studied about Progression(s) and their types which include Arithmetic Progression, Geometric Progression and Harmonic Progression. Further we have studied properties of these progressions. Formulas for finding n^{th} term, selections of n terms, sum of n terms for a given progression and insertion of n number of mean(s) between two given numbers are also studied.

Keywords:
AP; GP; HP; A_m; G_m; H_m; S_n

1. Introduction

A sequence is a function whose domain is the set *N* of natural numbers. If a_1, a_2, a_3, a_4,......a_n,..... is a sequence, then the expression $a_1+a_2+a_3+a_4+a_5+....+a_n+....$ is a series. A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

2. Progression

It is not necessarythat the terms of a sequence always follow a certain pattern or they are described by some explicit formula. Those sequences whose terms followcertain pattern are called progression.

2.1. ArithmeticProgression (AP)

A sequence is called an arithmeticprogression if the difference of a term and the previous term is always same, i.e., $a_{n+1}-a_n$=constant(=*d*),for each *n* belongs to *N*.

The constant difference, generally denoted by *d* is called the common difference.

e.g.

(1) 1,4,7,10,..... is an AP whose first term is 1 and common difference is 4 – 1=3.

(2) 11,7,3, -1,.... is an AP whose first term is 11 and the common difference is 7 – 11=-4.

2.1.1. Method to determine whether a Sequence is an AP or not when its *n*th term is given

a. Obtain a_n
b. Replace *n* by *n+1* in a_n to get a_{n+1}.
c. Calculate $a_{n+1}-a_n$.

If $a_{n+1}-a_n$is independent of *n*, the given sequence is an AP otherwise it is not an AP.

2.1.2. General Term of an AP

Let *a*be the first term and *d* be the common difference of an AP. Then the n^{th} term is a+(n-1)d.

i.e., $T_n=a+(n-1)d$

If*l* is the last term of a sequence, then

$$l=T_n=a+(n-1)d$$

2.1.3. *n*th term of an AP from the end

Let *a* be the first term and *d* be the common differenceof an AP having *m* terms. Then, n^{th} term from the end is $(m - n + 1)^{th}$ term from the beginning. So, nth term from the end is

$$T_{m-n+1}=a+(m-n+1-1)d=a+(m-n)d$$

2.1.4. Selection of terms in an AP

Sometimes we require selecting a certain numbers of terms in AP. The following ways of selecting terms are generally very convenient.

No. of terms	Terms	Common Difference
3	a-d, a, a+d	d
4	a-3d ,a-d, a+d,a+3d	2d
5	a-2d, a-d, a ,a+d ,a+2d	d
6	a-5d, a-3d, a-d, a+d, a+3d. a+5d	2d

2.1.5. Sum of *n* terms of an AP

The sum S_n of an AP with first term *a* and common difference *d* is

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

Or

$$S_n = \frac{n}{2}[a+l]$$

Where l=last term=a+(n-1)d

Also $T_n=S_n-S_{n-1}$

2.1.6. Insertion of *n*Arithmetic Means

Let $A_1, A_2, A_3,.....,A_n$ be *n* arithmetic means between two quantities *a* and *b*. Then, $a, A_1, A_2,...., A_n, b$ is an AP.

Let *d* be the common difference of this AP, Clearly it contains (n+2) terms

∴ $b=(n+2)^{th}$ term

⇨ b=a+(n+1)d

⇨ $d=\frac{b-a}{n+1}$

In generalm^{th}arithmetic mean is given by

$$A_m=a+\frac{m(b-a)}{n+1}$$

By putting m=1,2,3,....,n, we get $A_1, A_2,....,A_n$, which are requiredarithmetic means.

If *n*= 0, then $A_m = \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$

If there is only one arithmetic mean *A* between *a* and *b*, then a, A, b are in AP.

Then, $A = \frac{a+b}{2}$ here *n*=1

2.1.7. Properties of Arithmetic Progression

i. If a constant is added or subtracted from each term of an AP, Then the resulting sequence is also an AP with same common difference.

ii. If each term of a given AP is multiplied or divided by a non-zero constant *k*, then the resulting sequence is also an AP with common difference *Kd* or$\frac{k}{d}$, where *d* is the common difference of the given AP.

iii. A sequence is an AP iff its n^{th} term is of the form A_n+B i.e., a linear expression in *n*. The common difference in such a case is *A* i.e., the coefficient of *n*.

iv. In a finite AP the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.

i.e., $a_1+a_n = a_2 + a_{n-1} = a_3+a_{n-2} = ...$

v. If the terms of an AP are chosen at regular intervals then they form an AP.

vi. If p^{th} term of an AP is *q* and q^{th} term is *p*, then $T_{p+q} = 0$, $T_r = p + q - r$.

vii. If $S_p=S_q$ for an AP, then $S_{p+q} = 0$

viii. Sum of *n* A_m's between *a*and *b* is equals to *nA*.

2.2. Geometric Progression(GP)

A sequence of non-zero numbers is called a

Geometric Progression if the ratio of a term and the term proceeding to it is always a constant quantity.
The constant ratio, generally denoted by *r* is called the common ratio of the GP.
e.g., the sequence $\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}, -\frac{9}{8}$,..... is a GP with first term $\frac{1}{3}$ and common ratio$(-\frac{1}{2})/(\frac{1}{3})=(-\frac{3}{2})$.

2.2.1. General term of a GP

The n^{th} term of a GP with first term *a* and common ratio *r* is given by $a_n=ar^{n-1}$
GP can be written as: a, ar, ar^2,......,ar^{n-1},......
or a, ar, ar^2, ar^3, ar^4,......, ar^{n-1},....
according as it is finite or infinite.

2.2.2. The nth term from the end of a finite GP

The n^{th} term from the end of a finite GP consisting of *m* terms is ar^{m-n}, where *a* is the first term and *r* is the common ratio of the GP.

2.2.3. Selection of Terms in GP

Sometimes it is required to select a finite number of terms in GP. It is convenient if we select a term in following manner.

No. of terms	Terms	Common ratio
3	$\frac{a}{r}$ a, ar	r
4	$\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3	r^2
5	$\frac{a}{r^2}, \frac{a}{r}$, a, ar, ar^2	r

2.2.4. Sum of n terms of a GP

The sum of n terms of a GP with first term 'a' and common ratio 'r' is given by

$$S_n = a(\frac{1-r^n}{1-r}) \quad \text{for} \quad |r|<1$$

and

$$S_n = \frac{a(r^n-1)}{r-1} \quad \text{for} \quad |r|>1$$

The thing that must be noted is that the above formulae do not hold for *r* = 1. For *r* = 1, the sum of *n* terms of the GP is $S_n=na$.
If number of terms is infinite then the sum of the terms is

$$S=\frac{a}{1-r} \quad \text{for} \quad |r|<1$$

2.2.5. Insertion of n Geometric Means

Let G_1, G_2,.....,G_n be *n* geometric means between two quantities *a* and *b*, Then,a, G1, G2,......, Gn, b is a GP.
Clearly, it contains (n+ 2) terms. Let *r* be the common ratio of this GP.

∴ $b=(n+2)^{th}$ term$=ar^{r+1}$

⇨ $r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

In general m^{th} geometric mean is given by

$$G_m=a\left(\frac{b}{a}\right)^{\frac{m}{n+1}}$$

By putting *m*=1,2,3,.....,n, we get G_1, G_2,......, G_n. Which are the required geometric means.
If n=$-\frac{1}{2}$, then, $G_m= \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$
If there is only one geometric mean *G* between *a* and *b*, then *a*, *G*, *b* are in GP.
Then, $G_m=(ab)^{\frac{1}{2}}$

2.2.6. Properties of Geometric Progression

I. If all the terms of a GP be multiplied or divided by a non-zero constant, then it remains a GP with same common ratio.
II. The reciprocals of the terms of a given GP form a GP.
III. If each term of a GP be raised to same power, the resulting sequence also forms a GP.
IV. In a finite GP the product of terms equidistant from the beginning and the end is always same and is equal to the product of first and last term.
V. Three non-zero numbers *a*, *b* and *c*

are in GP, iff $b^2 = ac$.

VI. If the terms of a given GP are chosen at regular intervals, then the new sequence so formed also forms a GP.

VII. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a GP of non-zero, non-negative terms, then $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ is an AP and vice-versa.

VIII. If a, b and c are p^{th}, q^{th} and r^{th} terms of a GP respectively, then $a^{q-r}.b^{r-p}.c^{p-q} = 1$.

2.3. Harmonic Progression(HP)

A sequence $a_1, a_2, \ldots, a_n$ of non-zero numbers is called a Harmonic Progression, if the sequence of reciprocal of these numbers i.e., $\frac{1}{a1}, \frac{1}{a2}, \ldots\ldots, \frac{1}{an}$ is an AP.

e.g., the sequence $\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ is a HP because the sequence 1, 3, 5, 7,...... is an AP.

There is no formula for finding the sum of HP sequence.

2.3.1. General term of a HP

If the sequence $a_1, a_2, a_3, \ldots$ is a HP, then its n^{th} term

$$Tn = \frac{1}{\frac{1}{a1} + (n-1)\left(\frac{1}{a1} - \frac{1}{a2}\right)}$$

2.3.2. Insertion of *n* Harmonic Means

Let $H_1, H_2, \ldots, H_n$ be n harmonic means between two quantities a and b. Then, $a, H_1, H_2, \ldots, H_n, b$ is a HP.

Cleary, it contains (n+2) terms.

Also, $\frac{1}{a}, \frac{1}{H1}, \frac{1}{H2}, \ldots\ldots, \frac{1}{Hn}, \frac{1}{b}$ is an AP.

Let the common difference of this AP is d. Then,

$$\frac{1}{b} = (n+2)^{th} \text{ term}$$

$$\Rightarrow \frac{1}{b} = \frac{1}{a} + (n+1)d$$

$$\Rightarrow d = \frac{a-b}{(n+1)ab}$$

In general, m^{th} harmonic mean is given by

$$Hm = \frac{(n+1)ab}{ma + [n-(m-1)]b}$$

for, $m = 1, 2, \ldots, n$.

By putting $m = 1, 2, \ldots, n$, we get $H_1, H_2, \ldots, H_n$ which are required geometric means.

If $n = -1$, then, $Hm = \frac{a^{n+1} + b^{n+1}}{a^n + b^n}$

If there is only one harmonic mean H between a and b, then a, H, b are in HP.

Then, $H = \frac{2ab}{a+b}$

2.3.3. Properties of Harmonic Mean

i. If the m^{th} term of HP = n and n^{th} term = m, then $T_{m+n} = \frac{mn}{m+n}$ and $T_{mn} = 1$.

3. Relationship between Arithmetic, Geometric and Harmonic Means

Let A, G and H be the arithmetic, geometric and harmonic means between a and b, then

a. A >= G>=H
b. $G^2 = AH$
c. The equation having a and b as its roots is $x^2 - 2Ax + G^2 = 0$
d. If A_1, A_2 be two Am's; G_1, G_2 be two Gm's and H_1, H_2 be two Hm's between two numbers a and b, then, $\frac{G1G2}{H1H2} = \frac{A1 + A2}{H1 + H2}$.

4. Sum of *n* terms of special series

4.1. The sum of first n natural numbers

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

4.2. The sum of squares of the first n natural numbers

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

4.3. The sum of cubes of the first n natural numbers

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

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