

Study on Sets

Sujeet Kumar & Ashish Kumar Gupta

Department of Information and technology Dronacharya College of Engineering, Gurgaon-122001, India
Email:sujeet.16939@ggnindia.dronacharya.info, Email:ashish.16907@ggnindia.dronacharya.info

Abstract-

Set theory is the branch of mathematical logic that studies sets, which are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. In this research paper we studied about Basic concepts and notation, some ontology and applications. We have also study about combinational set theory, forcing, cardinal invariants, fuzzy set theory. We have described all the basic concepts of Set Theory.

Keywords-

Combinational ;fuzzy ; forcing; cardinals; ontology

1. INTRODUCTION

Set theory is the branch of mathematical logic that studies sets, which are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used in the definitions of nearly all mathematical objects.

1.1 HISTORY

Since the 5th century BC, beginning with Greek mathematician Zeno of Elea in the West and early Indian mathematicians in the East, mathematicians had struggled with the concept of infinity. Especially notable is the work of Bernard Bolzano in the first half of the 19th century.[3] Modern understanding of infinity began in 1867–71, with Cantor's work on number theory. An 1872 meeting between Cantor and Richard Dedekind influenced Cantor's thinking and culminated in Cantor's 1874 paper.

1.2 BASIC CONCEPTS

- **Union** of the sets A and B , denoted $A \cup B$, is the set of all objects that are a member of A , or B , or both. The union of $\{1, 2, 3\}$ and $\{2, 3, 4\}$ is the set $\{1, 2, 3, 4\}$.
- **Intersection** of the sets A and B , denoted $A \cap B$, is the set of all objects that are members of both A and B . The intersection of $\{1, 2, 3\}$ and $\{2, 3, 4\}$ is the set $\{2, 3\}$.

- **Set difference** of U and A , denoted $U \setminus A$, is the set of all members of U that are not members of A . The set difference $\{1, 2, 3\} \setminus \{2, 3, 4\}$ is $\{1\}$, while, conversely, the set difference $\{2, 3, 4\} \setminus \{1, 2, 3\}$ is $\{4\}$. When A is a subset of U , the set difference $U \setminus A$ is also called the **complement** of A in U . In this case, if the choice of U is clear from the context, the notation A^c is sometimes used instead of $U \setminus A$, particularly if U is a universal set as in the study of Venn diagrams.
- **Symmetric difference** of sets A and B , denoted $A \Delta B$ or $A \ominus B$, is the set of all objects that are a member of exactly one of A and B (elements which are in one of the sets, but not in both). For instance, for the sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$, the symmetric difference set is $\{1, 4\}$. It is the set difference of the union and the intersection, $(A \cup B) \setminus (A \cap B)$ or $(A \setminus B) \cup (B \setminus A)$.
- **Cartesian product** of A and B , denoted $A \times B$, is the set whose members are all possible ordered pairs (a, b) where a is a member of A and b is a member of B . The cartesian product of $\{1, 2\}$ and $\{\text{red}, \text{white}\}$ is $\{(1, \text{red}), (1, \text{white}), (2, \text{red}), (2, \text{white})\}$.
- **Power set** of a set A is the set whose members are all possible subsets of A . For example, the power set of $\{1, 2\}$ is $\{\{\}, \{1\}, \{2\}, \{1, 2\}\}$.

1.3 SOME ONTOLOGY

A set is pure if all of its members are sets, all members of its members are sets, and so on. For example, the set $\{\{\}\}$ containing only the empty set is a nonempty pure set. In modern set theory, it is common to restrict attention to the von Neumann universe of pure sets, and

many systems of axiomatic set theory are designed to axiomatize the pure sets only. There are many technical advantages to this restriction, and little generality is lost, because essentially all mathematical concepts can be modeled by pure sets. Sets in the von Neumann universe are organized into a cumulative hierarchy, based on how deeply their members, members of members, etc. are nested. Each set in this hierarchy is assigned (by transfinite recursion) an ordinal number α , known as its rank. The rank of a pure set X is defined to be the least upper bound of all successors of ranks of members of X . For example, the empty set is assigned rank 0, while the set $\{\{\}\}$ containing only the empty set is assigned rank 1. For each ordinal α , the set V_α is defined to consist of all pure sets with rank less than α . The entire von Neumann universe is denoted V .

1.4 AXIOMATIC SET THEORY

Elementary set theory can be studied informally and intuitively, and so can be taught in primary schools using [Venn diagrams](#). The intuitive approach tacitly assumes that a set may be formed from the class of all objects satisfying any particular defining condition. This assumption gives rise to paradoxes, the simplest and best known of which are Russell's paradox and the Burali-Forti paradox. Axiomatic set theory was originally devised to rid set theory of such paradoxes.

The most widely studied systems of axiomatic set theory imply that all sets form a cumulative hierarchy. Such systems come in two flavors, those whose ontology consists of:

- *Sets alone.* This includes the most common axiomatic set theory, **Zermelo–Fraenkel set theory (ZFC)**, which includes the axiom of choice. Fragments of ZFC include:
 - Zermelo set theory, which replaces the axiom schema of replacement with that of separation;
 - General set theory, a small fragment of Zermelo set theory sufficient for the Peano axioms and finite sets;

- Kripke–Platek set theory, which omits the axioms of infinity, powerset, and choice, and weakens the axiom schemata of separation and replacement.
- *Sets and proper classes.* These include Von Neumann–Bernays–Gödel set theory, which has the same strength as ZFC for theorems about sets alone, and Morse–Kelley set theory and Tarski–Grothendieck set theory, both of which are stronger than ZFC.

1.5 APPLICATIONS

Many mathematical concepts can be defined precisely using only set theoretic concepts. For example, mathematical structures as diverse as graphs, manifolds, rings, and vector spaces can all be defined as sets satisfying various (axiomatic) properties. Equivalence and order relations are ubiquitous in mathematics, and the theory of mathematical relations can be described in set theory.

Set theory is also a promising foundational system for much of mathematics. Since the publication of the first volume of *Principia Mathematica*, it has been claimed that most or even all mathematical theorems can be derived using an aptly designed set of axioms for set theory, augmented with many definitions, using first or second order logic. For example, properties of the natural and real numbers can be derived within set theory, as each number system can be identified with a set of equivalence classes under a suitable equivalence relation whose field is some infinite set.

Set theory as a foundation for mathematical analysis, topology, abstract algebra, and discrete mathematics is likewise uncontroversial; mathematicians accept that (in principle) theorems in these areas can be derived from the relevant definitions and the axioms of set theory. Few full derivations of complex mathematical theorems from set theory have been formally verified, however, because such formal derivations are often much longer than the natural language proofs mathematicians commonly present. One verification project, Metamath, includes

derivations of more than 10,000 theorems starting from the ZFC axioms and using first order logic.

1.6 AREAS OF STUDY

Combinatorial set theory

Main article: Infinitarycombinatorics

Combinatorial set theory concerns extensions of finite combinatorics to infinite sets. This includes the study of cardinal arithmetic and the study of extensions of Ramsey's theorem such as the Erdős–Rado theorem.

Descriptive set theory

Main article: Descriptive set theory

Descriptive set theory is the study of subsets of the real line and, more generally, subsets of Polish spaces. It begins with the study of pointclasses in the Borel hierarchy and extends to the study of more complex hierarchies such as the projective hierarchy and the Wadge hierarchy. Many properties of Borel sets can be established in ZFC, but proving these properties hold for more complicated sets requires additional axioms related to determinacy and large cardinals.

The field of effective descriptive set theory is between set theory and recursion theory. It includes the study of lightface pointclasses, and is closely related to hyperarithmetical theory. In many cases, results of classical descriptive set theory have effective versions; in some cases, new results are obtained by proving the effective version first and then extending ("relativizing") it to make it more broadly applicable.

A recent area of research concerns Borel equivalence relations and more complicated definable equivalence relations. This has important applications to the study of invariants in many fields of mathematics.

Fuzzy set theory[\[edit\]](#)

Main article: Fuzzy set theory

In set theory as Cantor defined and Zermelo and Fraenkel axiomatized, an object is either a member of a set or not.

In fuzzy set theory this condition was relaxed by Lotfi A. Zadeh so an object has a *degree of membership* in a set, a number between 0 and 1. For example, the degree of membership of a person in the set of "tall people" is more flexible than a simple yes or no answer and can be a real number such as 0.75.

Inner model theory[\[edit\]](#)

Main article: Inner model theory

An **inner model** of Zermelo–Fraenkel set theory (ZF) is a transitive [class](#) that includes all the ordinals and satisfies all the axioms of ZF. The canonical example is the constructible universe *L* developed by Gödel. One reason that the study of inner models is of interest is that it can be used to prove consistency results. For example, it can be shown that regardless of whether a model *V* of ZF satisfies the continuum hypothesis or the axiom of choice, the inner model *L* constructed inside the original model will satisfy both the generalized continuum hypothesis and the axiom of choice. Thus the assumption that ZF is consistent (has at least one model) implies that ZF together with these two principles is consistent.

The study of inner models is common in the study of determinacy and large cardinals, especially when considering axioms such as the axiom of determinacy that contradict the axiom of choice. Even if a fixed model of set theory satisfies the axiom of choice, it is possible for an inner model to fail to satisfy the axiom of choice. For example, the existence of sufficiently large cardinals implies that there is an inner model satisfying the axiom of determinacy (and thus not satisfying the axiom of choice).^{[\[5\]](#)}

Large cardinals

A **large cardinal** is a cardinal number with an extra property. Many such properties are studied, including inaccessible cardinals, measurable cardinals, and many more. These properties typically imply the cardinal number must be very large, with the existence of a cardinal with the specified property unprovable in Zermelo–Fraenkel set theory.

Determinacy

Determinacy refers to the fact that, under appropriate assumptions, certain two-player games of perfect information are determined from the start in the sense that one player must have a winning strategy. The existence of these strategies has important consequences in descriptive set theory, as the assumption that a broader class of games is determined often implies that a broader class of sets will have a topological property. The axiom of determinacy (AD) is an important object of study; although incompatible with the axiom of choice, AD implies that all subsets of the real line are well behaved (in particular, measurable and with the perfect set property). AD can be used to prove that the Wedge degrees have an elegant structure.

Forcing[\[edit\]](#)

Main article: Forcing (mathematics)

Paul Cohen invented the method of forcing while searching for a model of ZFC in which the continuum hypothesis fails, or a model of ZF in which the axiom of choice fails. Forcing adjoins to some given model of set theory additional sets in order to create a larger model with properties determined (i.e. "forced") by the construction and the original model. For example, Cohen's construction adjoins additional subsets of the natural numbers without changing any of the cardinal numbers of the original model. Forcing is also one of two methods for proving relative consistency by finitistic methods, the other method being Boolean-valued models.

Cardinal invariants[\[edit\]](#)

A **cardinal invariant** is a property of the real line measured by a cardinal number. For example, a well-studied invariant is the smallest cardinality of a collection of meagre sets of reals whose union is the entire real line. These are invariants in the sense that any two isomorphic models of set theory must give the same cardinal for each invariant. Many cardinal invariants have been studied, and the relationships between them are often complex and related to axioms of set theory.

Set-theoretic topology[\[edit\]](#)

Set-theoretic topology studies questions of general topology that are set-theoretic in nature or that require advanced methods of set theory for their solution. Many of these theorems are independent of ZFC, requiring stronger axioms for their proof. A famous problem is the normal Moore space question, a question in general topology that was the subject of intense research. The answer to the normal Moore space question was eventually proved to be independent of ZFC.

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