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Euler's Method for Fractional Differential Equations

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Abstract:

This paper presents a numerical method for solving fractional differential equations in the Riemann -Liouville sense. The approach is based on the Euler's method. The main characteristic behind the approach is that Euler method has intuitive geometric meaning. The algorithm is presented and the convergence of the algorithm is proved. As applications of main results, three specific numerical examples are given.

Keywords: Fractional Differential Equations, Initial Value Problem, Solution, Existence, Euler's Method.

1 Introduction:

With the rapid development of high-tech, the fractional calculus gets involved in more and more areas, especially in control theory – viscoelastic, theory-electronic, chemicals - fractal theory and so on. See reference [1]-[5]. The Existence and uniqueness for fractional differential equations has been investigated by many authors (see, e.g., [6]-[8]). Finding accurate and efficient methods for solving FDEs has been an active research undertaking. In the past few decades, many methods have been developed for solving FDEs from the numerical point of view, such as the Legendre wavelet method, the spectral method and quartered shifted Legendre method based on Gauss C.Labatt. See reference [9]-[11]. Euler's method has been proven to be efficient solving ordinary differential equations (ODEs) and other kinds of equations. See reference [12, 13]. A question arise naturally: can we have Euler method to derive numerical solution of FDEs? This paper is concerned with the numerical solution of following initial value problem of FDE

D^{α}_{a+} = f(x, y) Where $0 < \alpha < 1$ and fractional derivative is in Riemann-Liouville sense. In this paper, we give the Euler method for the fractional differential equations. This paper is organized as follows.

In section 2 we introduce some definitions and some relevant properties of Riemann-Liouville derivative and Caputo derivative. In section 3 we present the proof of convergence of the algorithm and error analysis of the algorithm. In section 4 improved algorithms are given. In section 5 we give three specific numerical examples equipped with comparing figure of numerical solution and analytical solution. Finally we conclude the paper with some remarks.

2 Preliminaries: There are a great number of definitions of fractional integration and fractional derivative (see, [14]-[17]). We will only present Riemann-Liouville and Caputo.

Let f : [a, b] → R be a function, α is a positive real number satisfying n − 1 ≤ α < n, and Γ the Euler gamma function.

Definition 1: The left and right Riemann-Liouville fractional integration of order α is defined by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt$$

Definition 2: The left and right Riemann-Liouville fractional derivative of order α is defined by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t) dt$$

and

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t) dt$$

Definition 3: The left and right Caputo fractional derivative of order α is defined by

$${}^C D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt$$

and

$${}^C D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (-1)^n (t-x)^{n-\alpha-1} f^{(n)}(t) dt$$

There exists a relation between the Riemann-Liouville fractional derivative and Caputo fractional derivative.

$$^{C}D^{\alpha}_{a+}f(x) = D^{\alpha}_{a+}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} \quad (1)$$

Lemma 4 *[18] Let $\alpha \geq 0$, $\beta \geq 0$, and $\varphi \in L^1[a,b]$, then*

$$I^{\alpha}_{a+}I^{\beta}_{a+}\varphi = I^{\alpha+\beta}_{a+}\varphi$$

Holds almost everywhere on [a ,b].

3. Euler's method to fractional differential equations and error analysis:

This paper is concerned with the numerical solution of following initial value problem of FDE.

$$D^{\alpha}_{a+}y = f(x,y) \quad (2)$$

$$y(a) = y_0 \quad (3)$$

The fractional derivative is in Riemann - Liouville sense with order $0<\alpha<1$ By using the properties of fractional integration and fractional derivative we can do analogously transformation as in this paper. If we apply Riemann- Liouville fractional derivative of 1- α order on (2) we get the following equation.

$$y' = D^{1-\alpha}_{a+}f(x,y) \quad (4)$$

According to the Euler's method we get the following algorathim:

$$\begin{cases} x_{n+1} &= x_0 + nh \\ h &= \frac{H}{n} \\ y_{n+1} - y_n &= hD^{1-\alpha}_{a+}f(x,y_n)|_{x=x_n} \end{cases} \quad (5)$$

With the Matlab software, the algorithm can be achieved in computer. And the algorithm is proved to be efficient and convergent. Before the proof, we will give some relevant definitions and Lemma.

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Definition 6 [19] Let $f_1(x)$, $f_2(x)$---,$f_n(x)$, be sequence of functions on interval I. It is called uniformly bounded if there exists a constant $K > 0$ such that $|f_n(x)| \leq K$ to all $x \epsilon I$ and $n \epsilon N$.

Definition 7 [19] Let $f_1(x)$, $f_2(x)$, _ _ _ , $f_n(x)$, _ _ _ be sequence of functions on interval I. It is called equicontinuous if arbitrary ε there exists δ such that for arbitrary x_1, $x_2 \epsilon I$ such that when

$|x_1 - x_2| < \delta$, $|f_n(x_1) - f_n(x_2)| < \varepsilon$ holds for all n.

Lemma 8 [19] Let $f_1(x)$, $f_2(x)$, _ _ _ , $f_n(x)$, _ _ _ be sequence of functions on finite closed interval I. I fit is uniformly bounded and equicontinuous, there is a subsequence which is uniformly continuous.

Lemma 9 [20] Function y has continuous left fractional derivative, then it is necessarily that

$y(a) = 0$

Theorem 10 Let the function f(x, y) satisfies conditions that $f(x_0, y(x_0)) = 0$ and fx(x, y) is continuous on R $0 \leq x - x_0 \leq c$, $|y - y_0| \leq b$

then the FDEs (2)–(3) have at least one solution at theinterval $0 \leq x - x_0 \leq H$ with

$H = \min \{ c, \frac{b}{M} \}$

$$M > \max_{(x,y)\in R} D_{a+}^{1-\alpha} f(x,y).$$

Proof: Divide the interval $0 \leq x - x0 \leq H$ into n equal parts. We can get $n + 1$ points:

$x_k = x_0 + \frac{kH}{n}$, $k = 0, 1, 2$ _ _ _ n.

From the initial point $P_0(x_0, y_0)$, we denote intersection point of the direction of $P_0(x_0, y_0)$ and vertical line $x = x_1$ as $P_1(x_1, y_1)$, line segment $[P_0, P_1]$ as the first Euler line. Successively we get the Euler line γ_n. For any x satisfying $0 \leq x - x_0 \leq H$ there exists an integer $0 \leq s \leq n - 1$ such that $x_s < x \leq x_{s+1}$. For each $n \epsilon N$, let $\{\varphi_n(x)\}$ denote the sequence:

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$$\varphi_n(x) = y_0 + \sum_{k=0}^{s-1} D_{a+}^{1-\alpha} f(x, y_k)|_{x=x_k}(x_{k+1} - x_k)$$
$$+ D_{a+}^{1-\alpha} f(x, y_s)|_{x=x_s}(x - x_s)$$
(6)

Since $|y - y_0| \leq b$, we have $\{\varphi_n(x) - y_0\} \leq b$, which implies $\{\varphi_n(x)\}$ is uniformly bounded. Due to $f(x_0, y(x_0)) = 0$ and equation (1), the Riemann- Liouville fractional derivatives is equal to the Caputo fractional derivative, i.e.,

$$D_{a+}^{1-\alpha} f(x, y) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \frac{d}{dt} f(t, y) dt$$
(7)

By continuity of $fx(x, y)$ and equation (7), the term $D_{a+}^{1-\alpha} f(x, y)$ is continuous,

and $D_{a+}^{1-\alpha} f(x, y)$ is bounded. So we have

$\{\varphi_n(s) - \varphi_n(t)\} \leq M(x - t)$

Namely, $\{\varphi_n(x)\}$ is equicontinuous. According Ascoli Lemma, we can choose a subsequence of Eulers function which is uniformly convergent at the interval $0 \leq x - x_0 \leq H$. Denote the chosen subsequence: $\varphi_{n1}, \varphi_{n2}, \_\_\_, \varphi_{nk}, \_\_\_$.

Let $F(x, y) = D_{a+}^{1-\alpha} f(x, y)$. We shall prove $\varphi_n(x) = y0 + \int_{x0}^{x} F(x, \varphi_n(x))dx + \delta_n(x)$

and $\delta_n(x) \to 0$. Noticing that

$F(x_i, y_i)(x_{i+1} - x_i) = \int_{xi}^{xi+1} F(x_i, y_i)dx$ we have

$F(x_i, y_i)(x_{i+1} - x_i) = \int_{xi}^{xi+1} F(x, \varphi_n(x)))dx + dn(i)$ where

$dn(i) = \int_{xi}^{xi+1} F\,[F(xi, yi) - F(x, \varphi n(x))]dx$ for $i = 0, 1, \_\_\_, s-1$.For $x_s < x \leq x_{s+1}$

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$F(x_s, y_s)(x - x_s) = \int_{xs}^{x} F(x, \varphi_n(x)))dx + d_n^*(x)$

And

$d_n^*(x) = \int_{xs}^{x} [F(x_s, y_s) - F(x, \varphi_n(x)]dx.$

Thus the identity (6) is equivalent to

$\varphi_n(x) = y_0 + \int_{x0}^{x} F(x, \varphi_n(x)dx \quad + \delta_n(x)$

where

$\delta_n(x) = \sum_{i=0}^{s-1} d_n(i) + d_n^*(x)$

According to the structure of Euler's method, we have

$|x - x_i| \leq \frac{H}{n} n$ and $|\phi_n(x) - y_i| \leq \frac{MH}{n}$

for $x_i < x\ x_{i+1}$. Since $D_{\alpha+}^{1-\alpha} f(x, y)$ is continuous, for arbitrary $\mathcal{E}$, there exists N such that for arbitrary $x_i < x \leq x_{i+1}$,

$|F(x_i, y_i) - F(x, \varphi_n(x))| < \frac{\mathcal{E}}{H}$. Therefore,

$d_n(i) \leq \int_{xi}^{xi+1} |F(x_i, y_i) - F(x, \varphi_n(x))|\, dx < \frac{\mathcal{E}}{n}$. And for $n > N$, $x_s < x \leq x_{s+1}$,

we have $|\delta_n^*(x| < \frac{\mathcal{E}}{n}$.

So when $n > N$, $|\delta_n(x)| < \frac{s\mathcal{E}}{n} + \frac{\mathcal{E}}{n} \leq \mathcal{E}$ namely, $\delta_n(x) \rightarrow 0$ Thus

$\varphi_{nk}(x) = y_0 + \int_{x0}^{x} F(x, \varphi_{nk}(x))\, dx + \delta_{nk}(x)$ Since the subsequence is uniformly convergent and

$\delta_n(x) \leq 0$, let $\phi(x)$ be the limit of $\{\varphi_{nk}(x)\}$, then

$\phi(x) = y_0 + \int_{x0}^{x} F(x, \varphi(x))\, dx$

for $0 \leq x - x_0 \leq H$. Hence the FDEs (2)–(3) have atleast one solution on $[x_0, x_0 + H]$.

Remark 11 *In addition, if $f_x (x, y)$ satisfies Lipschitz condition*

$|f_x (x, y_1) - f_x (x, y_2)| \leq L|y_1 - y_2|$,

Then the solution of FDEs (2)-(3) is unique.

Lemma 12 *If $fx(x, y)$ satisfies Lipschitz condition $|f_x (x, y_1) - f_x (x, y_2)| \leq L|y_1 - y_2|$,*

and conditions in Theorem 10 , then $F(x, y) = D_{\alpha+}^{1-\alpha} f(x, y)$ also satisfies Lipschitz condition for y.

Proof: Noting that

$$|F(x, y_1) - F(x, y_2)| = \frac{1}{\Gamma(\alpha)} \left| \int_a^x (x-t)^{\alpha-1}(f_x(x, y_1) - f_x(x, y_2))\, dt \right| \leq \frac{L}{\Gamma(\alpha)} |y1 - y2| \left| \int_a^x (x-t)^{\alpha-1} dt \right|$$

$$= \frac{L|(x-a)^\alpha|}{\alpha\Gamma(\alpha)}|y_1 - y_2|,$$

for $0 \leq x - x0 \leq c$, there exists d which satisfies $|(x - a)^\alpha| \leq d$. Hence

$$I\, F(x, y_1) - F(x, y_2) | \leq M\, I\, y_1 - y_2 I$$

Where $M = \frac{Ld}{\alpha\Gamma(\alpha)}$

Theorem 13 If $fx(x, y)$ satisfies Lipschitz condition $|fx(x, y1) - fx(x, y_2)| \leq L|y_1 - y_2|$

Then $y(x_n) - y_n = O(h)$.

Proof: The Euler iteration formula is based on $y_n = y(x_n)$, Then we can get

$$\bar{y}_{n+1} = y(x_n) + hD_{a+}^{1-\alpha} f(x, y_n)|_{x=x_n} \qquad (8)$$

So we can easily get

$$y(x_{n+1}) - \bar{y}_{n+1} = \frac{h^2}{2} y''(\xi)$$

Namely $, |y(x_{n+1}) - \vec{y}_{n+1}| \leq ch^2.$

According to equation (8) and Euler's iteration formula, we get

$$|\bar{y}_{n+1} - y_{n+1}|$$
$$\leq |y(x_n) - y_n|$$
$$+ h|(D_a^{1-\alpha} f(x_n, y(x_n)) - D_a^{1-\alpha} f(x_n, y_n)|$$
$$\leq (1 + hM)|y(x_n) - y_n|.$$

Hence,

$$|y(x_{n+1}) - y_{n+1}|$$
$$\leq |y(x_{n+1}) - \bar{y}_{n+1}| + |\bar{y}_{n+1} - y_{n+1}|$$
$$\leq (1 + hM)|y(x_n) - y_n| + ch^2.$$

Therefore, we have estimate

$$|e_{n+1}| \leq (1 + hM)|y(x_n) - y_n| + ch^2.$$

From above we can get the recursion formula

$$|e_n| \leq (1 + hM)^n |e_0| + \frac{ch}{M}[(1 + hM)^n - 1].$$

Since $x_n - x_0 = nh \leq H,$

$$(1 + hL)^n| \leq (e^{hL})^n \leq e^{HL} = g$$

At the same time we have $e = 0$. Consequently $|en| \leq \frac{ch}{M}(g-1)$, namely, $y(x_n) - y_n = O(h)$.

Remark 14 *Theorem 13 indicates the Euler's method*

4 Improved Euler's method

A question arises naturally: can we improve the ac- curacy of the algorithm? Firstly, we recall backward Euler method. It is as follows:

$y_{n+1} - y_n = h\, D^{1-\alpha}_{\alpha+} f(x, y_{n+1})\ |_{x=xn+1}$.

It is obvious that the backward Euler's algorithm is implicit. Euler's method and backward Euler's method have their own characteristics. Euler's method is much more convenient. But taking the numerical stability factors into account, backward Euler's method is often chosen. Backward Euler's equations are usually solved by iteration. And the essence of the iterative process is gradually explicit. The specific is:

$$\begin{cases} y_{n+1}^{(0)} - y_n = h \times D_{a+}^{1-\alpha} f(x, y_n)|_{x=x_{n+1}} \\ y_{n+1}^{(k+1)} - y_n = h \times D_{a+}^{1-\alpha} f(x, y_{n+1}^{(k)})|_{x=x_{n+1}} \\ k = 0, 1, 2, \cdots \end{cases}$$

By calculation, we can get local truncation error of the two methods

$$y(x_{n+1}) - y_{n+1} = \frac{h^2}{2} y''(\xi)$$

and

$$y(x_{n+1}) - y_{n+1} = -\frac{h^2}{2} y''(\xi)$$

It is easy to see that we can get higher accuracy method by the average of the two methods. By the average of the two methods, we get implicit trapezoidal method

y_{n+1} -y_n

$$= \frac{h}{2} \times D_{a+}^{1-\alpha} f(x, y_n)|_{x=x_n} + D_{a+}^{1-\alpha} f(x, y_n)|_{x=x_{n+1}}$$

which can be solved by iteration formula?

$$y_{n+1}^{(0)} - y_n = hD_{a+}^{1-\alpha} f(x, y_n)|_{x=x_{n+1}}$$

$$y_{n+1}^{(k+1)} - y_n = \frac{h}{2} D_{a+}^{1-\alpha} f(x, y_{n+1}^{(k)})|_{x=x_{n+1}}$$
$$+ D_{a+}^{1-\alpha} f(x, y_n)|_{x=x_n}$$

Although the trapezoidal method improves the accuracy, the algorithm is complex. In iterative formula, iteration operation is repeated several times which leads great amount of computation and difficultly to predict the results. In order to decrease the amount of computation, we hope the algorithm transferred to the next step calculation after only once or twice iteration operation. Therefore, we propose improved Euler's method

$$\begin{cases} x_{n+1} = x_0 + nh \\ y_p - y_n = hD_{a+}^{1-\alpha} f(x, y_n)|_{x=x_n} \\ y_c - y_n = hD_{a+}^{1-\alpha} f(x, y_p)|_{x=x_{n+1}} \\ y_{n+1} = \frac{1}{2}(y_p + y_c) \end{cases}$$

Next we prove the Euler method is effective with first order error

Theorem 15 If fx(x, y) satisfies Lipschitz condition

$|f_x (x, y_1) - f_x (x, y_2)| \leq L|y_1 - y_2|$,

We can get $y(x_n) - y_n = O(h^2)$ for improved Euler's method.

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Proof: Let Euler iteration formula be based on $y_n = y(x_n)$. We can get

Set $F(x,y) = D_{a+}^{1-\alpha} f(x,y)$. We can get

$$F(x_{n+1}, y_n + hy'(x_n)) = F(x_{n+1}, y(x_{n+1})) + F_y(x_{n+1}, \xi)(y_n + hy'(x_n) - y(x_{n+1})) \quad (10$$

$$F(x_{n+1}, y(x_{n+1})) = y'(x_n) + hy''(x_n) + \frac{h^2}{2}y'''(x_n) + \frac{h^3}{3!}y'''(x_n) + \cdots \quad (11)$$

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{3!}y''(x_n) + \cdots \quad (12)$$

By (9)-(11), we get

$$\bar{y}_{n+1} = y(x_n) + \frac{h}{2}(y'(x_n) + y'(x_n) + hy''(x_n) + \frac{h^2}{2}y'''(x_n) + \frac{h^2}{2}F_y(x_{n+1}, \xi)y''(x_n)) \quad (13)$$

Combining Eq. (12) and Eq. (13), we get $y(x_{n+1}) - \bar{y}_{n+1} = O(h^3)$, namely,

$|y(x_{n+1}) - \bar{y}_{n+1}| \le ch^2$. (14)

Let

$\varphi = \frac{1}{2}(F(x, y) + F(x + h, y + hF(x, y)).$

Then

$|\varphi(x, y, h) - \varphi(x, \bar{y}, h)| \leq \frac{1}{2} \;\; |F(x, y) - F(x, \bar{y})| + F(x + h, y + hF(x, y))$

$- F(x + h, \bar{y} + hF(x,\bar{y})) \;\; \leq M\,(1 + \frac{h}{2}\; M)\,|y - \bar{y}| \leq M\,(1 + \frac{h0}{2} M\,)\,|y - \bar{y}|$ and

$$\begin{aligned} & |\bar{y}_{n+1} - y_{n+1}| \\ \leq \;& |y(x_n) - y_n| + |\varphi(x_n, y(x_n), h - \varphi(x_n, y_n, , h)| \\ \leq \;& (1 + hL_\varphi)|y(x_n) - y_n|. \end{aligned}$$

Hence, we have

$$\begin{aligned} & |y(x_{n+1}) - y_{n+1}| \\ \leq \;& |y(x_{n+1}) - \bar{y}_{n+1}| + |\bar{y}_{n+1} - y_{n+1}| \\ \leq \;& (1 + hL_\varphi)|y(x_n) - y_n| + ch^3 \end{aligned}$$

Thus we get the recursion formula

$Ie_nI \;\leq (1 + hL_\varphi)^n \quad |e0| + \quad \frac{ch2}{L\varphi} \;\; ch^2 \;\; [(1 + hL_\varphi)^n - 1]$

By $x_n - x_0 = nh \leq H$, then

$(1 + hL_\varphi)^n \leq (e^{hL\varphi})^n \leq e^{HL\varphi} = g_\varphi$.

At the same time we have $e0 = 0$. Consequently, we get that

$|e_n| \leq \frac{ch2}{L\emptyset}\,(g_\varphi - 1)$, namely, $y(x_n) - y_n = O(h^2)$.

5 Examples

In this section, with the help of Matlab, we give two examples to illustrate the convergence of both Euler method and improved Euler method by comparison

figure of numerical solution under different segmentation and analytical solution.

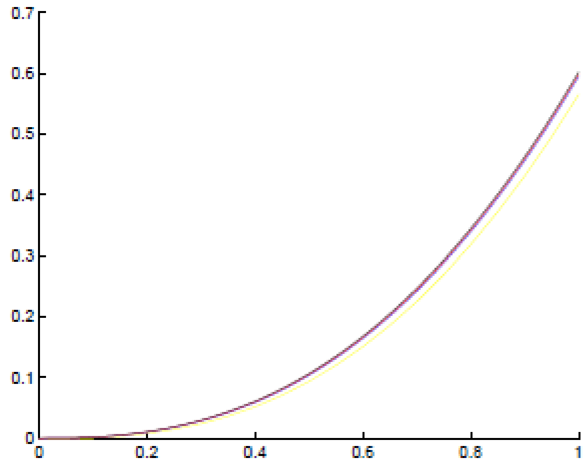

Figure 1:

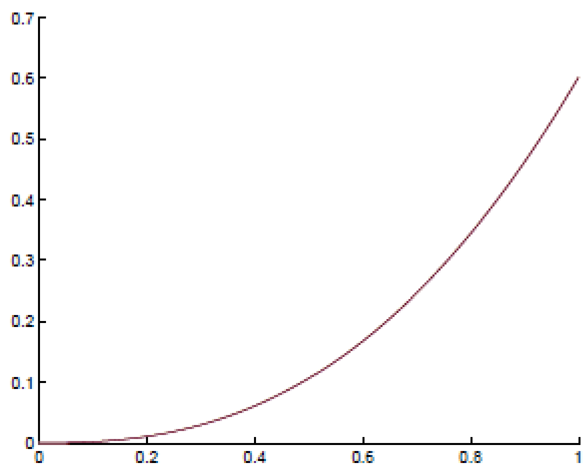

Figure 2:

Example 16 *Consider the following fractional differential equation [20]:*

$D_{0+}^{0.5} = x2,\ 0 \leq x\ \leq y(0) = 0$

The analytical solution is

$$y(x) = \frac{1}{\Gamma(0.5)} \frac{16x^{2.5}}{15}$$

For each method, we get three sets of numerical solution when the number of division is $n = 20$,

For each method, we get three sets of numerical solution when the number of division is $n = 20$, $n = 100$ and $n = 200$ respectively. Using the Matlab software, we get Fig.1 for Euler's method and Fig.2 for improved Euler's method. In the both figures, the yellow, blue, red and black curve are corresponding to numerical solution of $n = 20$, 100, 200 and the analytical solution respectively.

From the above two figures, we see that the numerical solution is closer to analytical solution as the number of division increase for the same method;

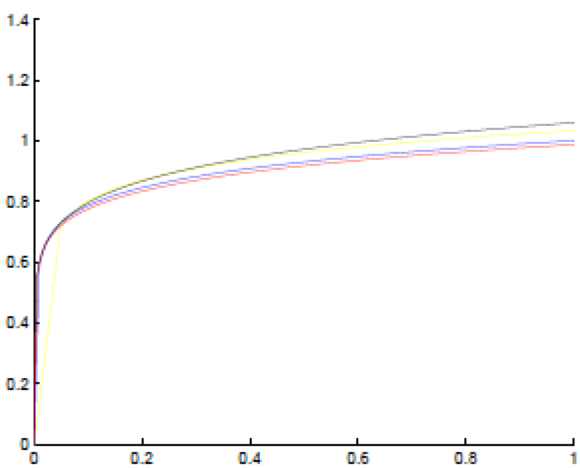

Figure 3:

numerical solution of improved Euler method is closer to analytical solution than the numerical solution of Euler method.

In this example Euler's method is enough perfect. In actual use, with improved Euler's method more stable and accurate but Euler's method operation on the computer is faster; we can choose a more suitable upon request method.

Example 17 *Consider the following fractional differential equation [21]:*

$$D_{0+}^{0.1}y = y^{0.2}, \quad 0 \leq x \leq 1$$

$y(0) = 0.$

The analytical solution is

$$y(x) = (\frac{\Gamma(1.125)}{\Gamma(1.025)})^{-1.25}x^{0.125}.$$

Using the MATLAB software, we get Fig.3 for Euler method and Fig.4 for improved Euler method. The yellow, blue, red and black curve are corresponding

to numerical solution of n = 20, 100, 200 and the analytical solution respectively.

According to the above two figures, the numerical solution appear larger deviation using Euler's method but not made up by improved Euler's method when n = 20 at the beginning certain number of points. This is to say the improved Euler's method is more accurate.

Example 18 *Consider the following fractional differential equation:*

$$D_{0+}^{0.5}y = \frac{5x^2}{6} + \frac{1}{6}(\frac{15\Gamma(0.5)y}{16})^{0.8}, \quad 0 \leq x \leq 1$$

$y(0) = 0$

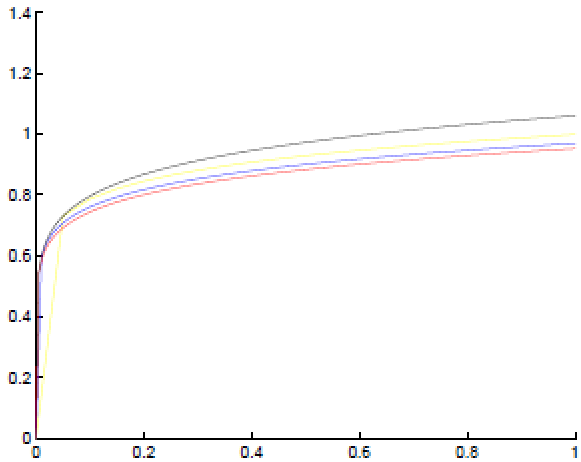

Figure 4:

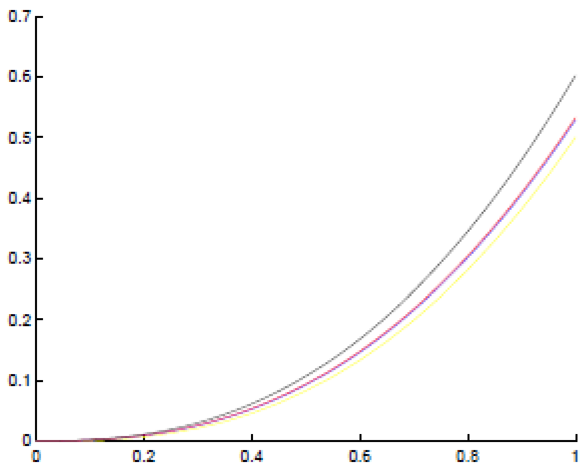

Figure 5:

The analytical solution is

$$y(x) = \frac{1}{\Gamma(0.5)} \frac{16x^{2.5}}{15}.$$

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For each method, we get three sets of numerical solution for the number of division is $n = 20$, $n = 100$ and $n = 200$ respectively. Using the Matlab software, we get Fig.5 for Euler method and Fig.6 for improved Euler method. In the two figure, the yellow, blue, red and black curve are corresponding to numerical solution of $n = 20, 100, 200$ and the analytical solution respectively.

6 Conclusion:

In this paper we derive a simple numerical method, Euler's method for solving fractional differential equations in the Riemann-Liouville sense, which has intuitive geometric meaning. And the numerical solution is closer to analytical solution as the number

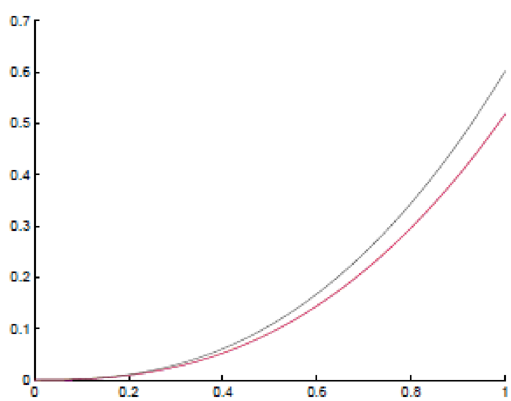

Figure 6:

of division increase. In actual use, when more stable and accurate is needed, we considered improve the method, which brings improved Euler's method. Compared with other algorithms, the algorithms in this paper are easier to understand and more simple to be operated on the computer. In this paper we only consider the fractional derivatives in Riemann-Liouville sense with the order $0 < \alpha < 1$, it can be generalized to any other order and fractional derivatives in other sense by using the relationship among various fractional derivatives. Acknowledgements: We sincerely acknowledge the anonymous reviewers for their comments. This research is supported by the Doctoral Fund of Education Ministry of China(20134219120003); The Natural Science Foundation of Hubei Province (2013CFA131) and Hubei Province Key Laboratory of Systems Science in Metallurgical Process(z201302).

References:

[1] J. T. Machado, V. Kiryakova , F. Mainardi, Recent history of fractional calculus, Commun. Nonlinear. Sci. Numer. Simulat. 16, 2011, pp. 1140–1153.

[2] G. Cooper, D. Cowan, The application of fractional calculus to potential field data, Explor. Geophys. 34, 2003, pp. 51–56.

[3] Z. E. A. Fellah, C. Depollier, M. Fellah, Application of fractional calculus to the sound

waves propagation in rigid porous materials: Validation via ultrasonic measurements, Acta. Acust. united. Ac. 88, 2002, pp. 34–39.

[4] J. Sabatier, O. P. Agrawal, J.A. Machado, Advances in fractional calculus: Theoretical devel-

WSEAS TRANSACTIONS on MATHEMATICS Ping Tong, Yuqiang Feng, Hengjin Lv

E-ISSN: 2224-2880 1152 Issue 12, Volume 12, December 2013 opments and applications in physics and engineering, Springer Publishing Company 2007

[5] F. Mainardi, Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models, Imperial College Press 2010.

[6] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, Nonlinear. Analy. Theor. 72, 2010, pp. 2859–2862.

[7] R. P. Agarwal, D. O'Regan, S. Stan¨ek, Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations, J. Math. Anal. Appl. 371, 2010, pp. 57–68.

[8] A. Yakar, M. E. Koksal, Existences Results for Solutions of Nonlinear Fractional Differential Equations, Abstr. Appl. Anal. 2012

[9] M. U. Rehman, R. A. Khan , The Legendre wavelet method for solving fractional differential equations, Commun. Nonlinear. Sci. Numer. Simulat. 16, 2011, pp. 4163–4173.

[10] E. H. Doha, A. H. Bhrawy , S. S. Ezz-Eldien, Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations, Appl. Math. Model. 35, 2011, pp. 5662– 5672.

[11] A. H. Bhrawy, A. S. Alofi, S. S. Ezz-Eldien, A quadrature tau method for fractional differential equations with variable coefficients, Appl. Math. Lett. 24, 2011, pp. 2146–2152.

[12] D. J. Higham, X. Mao, A. M. Stuart, Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM.J. Numer. Analy. 40, 2002, pp. 1041–1063.

[13] Z. Fan, M. Liu, W. Cao, Existence and uniqueness of the solutions and convergence of semiimplicit Euler methods for stochastic

pantograph equations, J. Math. Analy. Appl. 325, 2007,

pp. 1142–1159.

[14] V. Lakshmikantham, A. S. Vatsala, Basic theory of fractional differential equations, Nonlinear. Analy. Theor. 69, 2008, pp. 2677–2682.

[15] S. Das, Functional fractional calculus, Springer –Verlag, Berlin–Heidelberg 2011.

[16] S. G. Samko, A. A. Kilbas, O. O. I. Marichev, Fractional integrals and derivatives, Yverdon: Gordon and Breach Science Publishers 1993.

[17] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl. 265, 2002, pp. 229–248.

[18] T. J. Xiao, J. Liang, Approximations of Laplace transforms and integrated semigroups, J. Func. Anal. 172, 2000, pp. 202–220.

[19] L. Nirenberg, Topics in nonlinear functional analysis, American Mathematical Society 2001.

[20] R. Almeida, R. A. C. Ferreira, D. F. M. Torres, soperimetric problems of the calculus of variations with fractional derivatives, Acta. Math. Sci. 32, 2012, pp. 619–630.