Introduction to Relations

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ABSTRACT

This paper introduces you to relation, the word "relation" suggests some common examples of relation in world like relation of sun from planets like relation of a mother to her son and so on, likewise there are common relation in arithmetic such as greater than, less than or equality between two numbers, we also know the relation between the area of square and its sides, relation of circle and its radius and therefore these are the simple examples of relation between two different things or objects similarly we can have a relation among four or more objects.

INTRODUCTION

A relation is any association between elements of one set, called the domain or set of inputs, and another set, called the range or set of outputs. Some people mistakenly refer to the range as the co-domain (range), but as we will see, that really means the set of all possible outputs—even values that the relation does not actually use. (Beware: some authors do not use the term co-domain (range), and use the term range instead for this purpose. Those authors use the term image for what we are calling range. So while it **is** a mistake to refer to the range or image as the co-domain (range), it is not necessarily a mistake to refer to co-domain as range.

For example, if the domain is a set Fruits = {apples, oranges, bananas} and the co-domain (range) is a set Flavors = {sweetness, tartness, bitterness}, the flavors of these fruits form a relation: we might say that apples are related to (or associated with) both sweetness and tartness, while oranges are related to tartness only and bananas to sweetness only. (We might disagree somewhat, but that is irrelevant to the topic of this book.) Notice that "bitterness", although it is one of the possible Flavors (co-domain) (range), is not really used for any of these relationships; so it is not part of the range (or image) {sweetness, tartness}.

Another way of looking at this is to say that a relation is a subset of ordered pairs drawn from the set of all possible ordered pairs (of elements of two other sets, which we normally refer to as the Cartesian product of those sets). Formally, R is a relation if

$$R \subseteq X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

for the domain X and co-domain (range) Y. The inverse relation of R, which is written as R^{-1}, is what we get when we interchange the X and Y values:

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}$$

Using the example above, we can write the relation in set notation: {(apples, sweetness), (apples, tartness), (oranges, tartness), (bananas, sweetness)}. The inverse relation, which we could describe as "fruits of a given flavor", is {(sweetness, apples), (sweetness, bananas), (tartness, apples), (tartness, oranges)}. (Here, as elsewhere, the order of elements in a set has no significance.)

Notations

When we have the property that one value is related to another, we call this relation a binary relation and we write it as

x R y

where R is the relation.

For arrow diagrams and set notations, remember for relations we do not have the restriction that functions do and we can draw an arrow to represent the mappings, and for a set diagram, we need only write all the ordered pairs that the relation does take: again, by example

f = {(0,0),(1,1),(1,-1),(2,2),(2,-2)}

is a relation and not a function, since both 1 and 2 are mapped to two values, 1 and -1, and 2 and -2 respectively) example let A=2,3,5;B=4,6,9 then A*B=(2,4),(2,6),(2,9),(3,4),(3,6),(3,9),(5,4),(5,6),(5,9) Define a relation R=(2,4),(2,6),(3,6),(3,9) add functions and problems to one another

examples

Say f is defined by

{(0,0),(1,1),(2,2),(3,3),(1,2),(2,3),(3,1),(2,1),(3,2),(1,3)}

This is a relation (not a function) since we can observe that 1 maps to 2 and 3, for instance.

Less-than, "<", is a relation also. Many numbers can be less than some other fixed number, so it cannot be a function.

Properties

When we are looking at relations, we can observe some special properties different relations can have.

Reflexive

A relation is reflexive if, we observe that for all values a:

a R a

In other words, all values are related to themselves.

The relation of equality, "=" is reflexive. Observe that for, say, all numbers a (the domain is **R**):

a = a

so "=" is reflexive.

In a reflexive relation, we have arrows for all values in the domain pointing back to themselves:

Note that ≤ is also reflexive (a ≤ a for any a in **R**). On the other hand, the relation < is not (a < a is false for any a in **R**).

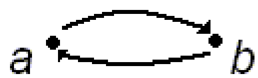
Symmetric

A relation is symmetric if, we observe that for all values of a and b:

a R b implies b R a

The relation of equality again is symmetric. If x=y, we can also write that y=x also.

In a symmetric relation, for each arrow we have also an opposite arrow, i.e. there is either no arrow between x and y, or an arrow points from x to y and an arrow back from y to x:



Neither $\leq$ nor $<$ is symmetric ($2 \leq 3$ and $2 < 3$ but neither $3 \leq 2$ nor $3 < 2$ is true).

Transitive

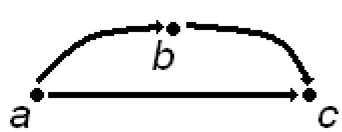
A relation is transitive if for all values a, b, c:

a R b and b R c implies a R c

The relation greater-than ">" is transitive. If $x > y$, and $y > z$, then it is true that $x > z$. This becomes clearer when we write down what is happening into words. x is greater than y and y is greater than z. So x is greater than both y and z.

The relation is-not-equal "$\neq$" is not transitive. If $x \neq y$ and $y \neq z$ then we might have $x = z$ or $x \neq z$ (for example $1 \neq 2$ and $2 \neq 3$ and $1 \neq 3$ but $0 \neq 1$ and $1 \neq 0$ and $0 = 0$).

In the arrow diagram, every arrow between two values a and b, and b and c, has an arrow going straight from a to c.



Anti-symmetric

A relation is anti-symmetric if we observe that for all values a and b:

a R b and b R a implies that a=b

Notice that anti-symmetric is not the same as "not symmetric."

Take the relation greater than or equal to, "$\geq$" If $x \geq y$, and $y \geq x$, then y must be equal to x. a relation is anti-symmetric if and only if $a \in A$, $(a,a) \in R$

Trichotomy

A relation satisfies trichotomy if we observe that for all values a and b it holds true that: aRb or bRa

The relation is-greater-or-equal satisfies since, given 2 real numbers a and b, it is true that whether $a \geq b$ or $b \geq a$ (both if $a = b$).

Equivalence relations

We have seen that certain common relations such as "=", and congruence (which we will deal with in the next section) obey some of these rules above. The relations we will deal with are very important in discrete mathematics, and are known as equivalence relations. They essentially assert some kind of equality notion, or equivalence, hence the name.

Characteristics of equivalence relations

For a relation R to be an equivalence relation, it must have the following properties, viz. R must be:

- symmetric
- transitive
- reflexive

(A helpful mnemonic, S-T-R)

In the previous problem set you have shown equality, "=", to be reflexive, symmetric, and transitive. So "=" is an equivalence relation.

We denote an equivalence relation, in general, by $x \sim y$.

Example proof

Say we are asked to prove that "=" is an equivalence relation. We then proceed to prove each property above in turn (Often, the proof of transitivity is the hardest).

- **Reflexive**: Clearly, it is true that a = a for all values a. Therefore, = is reflexive.
- **Symmetric**: If a = b, it is also true that b = a. Therefore, = is symmetric
- **Transitive**: If a = b and b = c, this says that a is the same as b which in turn is the same as c. So a is then the same as c, so a = c, and thus = is transitive.

Thus = is an equivalence relation.

Partial orders

We also see that "≥" and "≤" obey some of the rules above. Are these special kinds of relations too, like equivalence relations? Yes, in fact, these relations are specific examples of another special kind of relation which we will describe in this section: the partial order.

As the name suggests, this relation gives some kind of ordering to numbers.

Characteristics of partial orders

For a relation R to be a partial order, it must have the following three properties, viz R must be:

- reflexive
- antisymmetric
- transitive

(A helpful mnemonic, R-A-T)

We denote a partial order, in general, by $x \preceq y$.

Posets

A partial order imparts some kind of "ordering" amongst elements of a set. For example, we only know that 2 ≥ 1 because of the partial ordering ≥.

We call a set A, ordered under a general partial ordering $\preceq$, a partially ordered set, or simply just poset, and write it (A, $\preceq$).

Terminology

There is some specific terminology that will help us understand and visualize the partial orders.

When we have a partial order $\preceq$, such that a $\preceq$b, we write $\prec$to say that a $\preceq$but a ≠ b. We say in this instance that a precedes b, or a is a predecessor of b.

If (A, $\preceq$) is a poset, we say that a is an immediate predecessor of b (or a immediately precedes b) if there is no x in A such that a $\prec$x $\prec$b.

If we have the same poset, and we also have a and b in A, then we say a and b are comparable if a $\preceq$b or b $\preceq$a. Otherwise they are incomparable.

Hasse diagrams

Hasse diagrams are special diagrams that enable us to visualize the structure of a partial ordering. They use some of the concepts in the previous section to draw the diagram.

A Hasse diagram of the poset (A, $\preceq$) is constructed by

- placing elements of A as points
- if a and b ∈ A, and a is an immediate predecessor of b, we draw a line from a to b
- if a $\prec$b, put the point for a lower than the point for b
- not drawing loops from a to a (this is assumed in a partial order because of reflexivity)

Operations on Relations

There are some useful operations one can perform on relations, which allow to express some of the above mentioned properties more briefly.

Inversion

Let R be a relation, then its inversion, R^{-1} is defined by

R^{-1} := {(a,b) | (b,a) in R}.

Concatenation

Let R be a relation between the sets A and B, S be a relation between B and C. We can concatenate these relations by defining

R • S := {(a,c) | (a,b) in R and (b,c) in S for some b out of B}

Diagonal of a Set

Let A be a set, then we define the diagonal (D) of A by

D(A) := {(a,a) | a in A}

Shorter Notations

Using above definitions, one can say (lets assume R is a relation between A and B):

R is transitive if and only if R • R is a subset of R.

R is reflexive if and only if D(A) is a subset of R.

R is symmetric if R^{-1} is a subset of R.

R is antisymmetric if and only if the intersection of R and R^{-1} is D(A).

R is asymmetric if and only if the intersection of D(A) and R is empty.

R is a function if and only if R^{-1} • R is a subset of D(B).

In this case it is a function A → B. Let's assume R meets the condition of being a function, then

R is injective if R • R^{-1} is a subset of D(A).

R is surjective if {b | (a,b) in R} = B

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