International Journal of Research
Available at https://edupediapublications.org/journals
e-ISSN: 2348-6848
p-ISSN: 2348-795X
Volume 02 Issue 12
December 2015

Mathematics of Fluid & Its Applications

Versha Chopra
Assistant Professor in Mathematics
Vidyavati Anand DAV College for Women
Karnal (Haryana)

ABSTRACT: *The aim of this Paper is to furnish some results in very different areas that are linked by the common scope of giving new insight in the field of fluid dynamics. Thus, already by the time of the Roman Empire enough practical information had been accumulated to permit quite sophisticated applications of fluid dynamics.The study of these flows has been attached with a wide range of mathematical techniques and, to day, this is a stimulating part of both pure an applied mathematics. Fluid is a material that is infinitely deformable or malleable. A fluid may resist moving from one shape to another but resists the same amount in all directions and in all shapes. The basic characteristic of the fluid is that it can flow. Fluids are divided in two categories, incompressible fluids (fluids that move at far subsonic speeds and do not change their density) and compressible fluids.*

KEYWORDS: Fluid Dynamics, Applications, Mathematical, Techniques, Practical.

INTRODUCTION: The introducing "intuitive notion" of what constitutes a fluid. As already indicated we are accustomed to being surrounded by fluids both gases and liquids are fluids and we deal with these in numerous forms on a daily basis [1-4]. As a consequence, we have a fairly good intuition regarding what is, and is not, a fluid; in short we would probably simply say that a fluid is "anything that flows." This is actually a good practical view to take, most of the time. But we will later see that it leaves out some things that are fluids, and includes things that are not. So if we are to accurately analyze the behavior of fluids it will be necessary to have a more precise definition [5-8]. It is interesting to note that the formal study of fluids began at least 500 hundred years ago with the work of Leonardo da Vinci, but obviously a basic practical understanding of the behavior of fluids was available much earlier, at least by the time of the ancient Egyptians; in fact, the homes of well-to-do Romans had flushing toilets not very diff erent from those in modern 21-Century houses, and the Roman aqueducts are still considered a tremendous engineering feat [9]. Thus, already by the time of the Roman Empire enough practical information had been accumulated to permit quite sophisticated applications of fluid dynamics.

LITERATURE REVIEW:

The following example will provide a simple illustration of how to apply the above-stated principle. It will be evident that little is required beyond making direct use of its contents. We consider a cubical object with sides of length h that is floating in water in such a way that $\frac{1}{4}h$ of its vertical side is above the surface of the water, as indicated in Fig. 1. It is required to find the density of this cube.

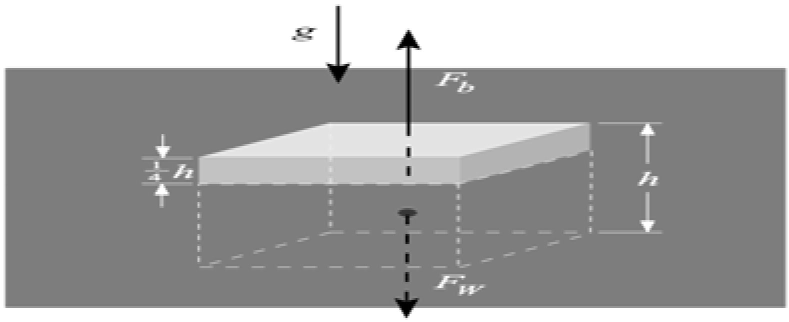

Figure 1: Application of Archimedes' principle to the case of a floating object.

From Archimedes' principle it follows directly that the buoyancy force must be

$$F_b = \rho_{water}\frac{3}{4}h^3 g\,,$$

And it must be pointing upward as indicated in the figure. The force due to the mass of the cube, i.e., the weight, is given by

$$F_W = -\rho_{cube}h^3 g\,,$$

With the minus sign indicating the downward direction of the force Now in static equilibrium (the cube floating as indicated), the forces acting on the cube must sum to zero. Hence,

$$F_b + F_W = 0 = \rho_{water}\frac{3}{4}h^3 g - \rho_{cube}h^3 g\,,$$

Or

$$\rho_{cube} = \frac{3}{4}\rho_{water}\,,$$

1. Existence of an approximate solution: Now we shall be concerned with the existence of a solution to the discrete Stokes problem [13]. For simplicity we consider the case with zero boundary condition.

Theorem

1. $((\boldsymbol{v}_h, \boldsymbol{v}_h))_h^{1/2} = |||\boldsymbol{v}_h|||_h$ *is a norm in* $\boldsymbol{X}_{h0}$,

2. BB condition holds:

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{X}_{h0}} \frac{(q_h, \operatorname{div}_h \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_h} \geq \gamma \|q_h\| \quad \forall\, q_h \in M_{h0}$$

Then the discrete Stokes problem has a unique solution $\boldsymbol{u}_h, p_h$.

Proof We want to find $\boldsymbol{u}_h \in \boldsymbol{X}_{h0},\ p_h \in M_{h0}$ satisfying

$$(+) \qquad \nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (p_h, \operatorname{div}_h \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_{h0}$$

$$(*) \qquad -q(h, \operatorname{div}_h \boldsymbol{u}_h) = 0 \quad \forall\, q_h \in M_{h0}.$$

We define

$$\boldsymbol{V}_h = \{\boldsymbol{v}_h \in \boldsymbol{X}_{h0}; (q_h, \operatorname{div}_h \boldsymbol{v}_h) = 0 \ \forall q_h \in M_{h0}\}$$

Is a continuous linear functional on $\boldsymbol{X}_{h0}$ and there exists $\tilde{B}q_h \in \boldsymbol{X}_{h0}$ such that

$$((\tilde{B}q_h, \boldsymbol{v}_h))_h = (q_h, \operatorname{div}_h \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_{h0}.$$

By the BB condition,

$$\sup_{0 \neq \boldsymbol{v}_h \in \boldsymbol{X}_{h0}} \frac{((\tilde{B}q_h, \boldsymbol{v}_h))_h}{|||\boldsymbol{v}_h|||_h} \geq \gamma \|q_h\| \quad \forall q_h \in M_{h0}.$$

We see that

$$\tilde{B}: \ M_{h0} \to \boldsymbol{X}_{h0} \quad \text{is a linear operator and}$$
$$|||\tilde{B}q_h|||_h \geq \gamma \|q_h\| \quad \forall q_h \in M_{h0}.$$

for $\tilde{F}$ there exists a unique $p_h \in M_{h0}$ such that $\tilde{B}p_h = \tilde{F}$. This is equivalent to the relation

$$((\tilde{B}p_h, \boldsymbol{v}_h))_h = ((\tilde{F}, \boldsymbol{v}_h))_h \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_{h0},$$

which means that

$$(p_h, \operatorname{div}_h \boldsymbol{v}_h) = -F(\boldsymbol{v}_h) = +\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_{h0},$$

and, thus,

$$\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (p_h, \operatorname{div}_h \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_{h0},$$

Which we wanted to prove

2. Choice of the finite element spaces:

We define $\boldsymbol{X}_h$, M_h as spaces of piecewise polynomial functions. However, cannot be chosen in an arbitrary way.

Example

$$\begin{aligned}
X_h &= \{v_h \in C(\overline{\Omega}); v_h|K \in P^1(K) \ \forall K \in \mathcal{T}_h\}, \\
X_{h0} &= \{v_h \in X_h; v_h|_{\partial\Omega} = 0\} \\
\boldsymbol{X}_h &= X_h \times X_h \\
\boldsymbol{X}_{h0} &= X_{h0} \times X_{h0} \\
\boldsymbol{V}_h &= \{v_h \in \boldsymbol{X}_{h0}; \operatorname{div}(v_h|_K) = 0 \ \forall K \in \mathcal{T}_h\}
\end{aligned}$$

This is in agreement with the above definition of V_h, provided

International Journal of Research
Available at https://edupediapublications.org/journals
e-ISSN: 2348-6848
p-ISSN: 2348-795X
Volume 02 Issue 12
December 2015

$$M_h = \{q \in L^2(\Omega); q \ \text{ is constant on each } K \in \mathcal{T}_h\},$$
$$M_{h0} = \left\{q \in M_h; \int_\Omega q\, dx = 0\right\}.$$

In this case we have

$$((\cdot,\cdot))_h = ((\cdot,\cdot)), \quad |||\cdot|||_h = |||\cdot|||$$
$$b_h = b, \quad \operatorname{div}_h = \operatorname{div}$$

Let $\Omega = (0,1)^2$, $\boldsymbol{u}_h \in \boldsymbol{V}_h$. Then

$$\boldsymbol{u}_h|_{\partial\Omega} = 0, \quad \operatorname{div}(\boldsymbol{v}_h|_K) = 0 \quad \forall K \in \mathcal{T}_h.$$

3- Bernoulli's Equation: Bernoulli's equation is one of the best-known and widely-used equations of elementary fluid mechanics [14]. In the present section we will derive this equation from the N.–S. equations again emphasizing the ease with which numerous seemingly scattered results can be obtained once these equations are available [10-12]. Following this derivation we will consider two main examples to highlight applications.

4. Continuous problem:

अ **Stokes problem:**

Let $\varphi \in \boldsymbol{H}^{1/2}(\partial\Omega)$, $\int_{\partial\Omega} \varphi \cdot \boldsymbol{n}\, ds = 0$, $\boldsymbol{g} \in \boldsymbol{H}^1(\Omega)$, $\operatorname{div} \boldsymbol{g} = 0$ in Ω, $\boldsymbol{g}|_{\partial\Omega} = \varphi$, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega), \boldsymbol{u}^* \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{u}^*|_{\partial\Omega} = \varphi$ and consider the Stokes problem to find $\boldsymbol{u}$ such that

$$\boldsymbol{u} - \boldsymbol{g} \in \boldsymbol{V},$$
$$\nu((\boldsymbol{u},\boldsymbol{v})) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

This can be reformulated with the aid of the pressure: Find u, p such that

$$\boldsymbol{u} - \boldsymbol{u}^* \in \boldsymbol{H}_0^1(\Omega), \quad p \in L_0^2 = \left\{q \in L^2(\Omega); \int_\Omega q\, dx = 0\right\},$$
$$\nu((\boldsymbol{u},\boldsymbol{v})) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),$$
$$-(q, \operatorname{div} \boldsymbol{u}) = 0 \qquad \forall q \in L_0^2(\Omega).$$

Exercise-1

Let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{u}|_{\partial\Omega} = \varphi$, $\int_{\partial\Omega} \varphi \cdot \boldsymbol{n}\, dS = 0$. Prove that $\operatorname{div} \boldsymbol{u} = 0$

$$-\ (q, \operatorname{div} \boldsymbol{u}) = 0 \quad \forall q \in L^2(\Omega) \quad (+) \quad \Leftrightarrow$$
$$-\ (q, \operatorname{div} \boldsymbol{u}) = 0 \quad \forall q \in L_0^2(\Omega). \quad (*)$$

Proof The implication $\Rightarrow$ is obvious.

The implication $\Leftarrow$: Let (*) hold. Then we can write

$$q \in L^2(\Omega) \to \tilde{q} = q - \frac{1}{|\Omega|}\int_\Omega q\, dx \in L_0^2(\Omega) \Rightarrow$$
$$\Rightarrow 0 = (\tilde{q}, \operatorname{div} \boldsymbol{u}) = (q, \operatorname{div} \boldsymbol{u}) - \frac{1}{|\Omega|}\int_\Omega q\, dx (1, \operatorname{div} \boldsymbol{u})$$

$$(1, \operatorname{div} \boldsymbol{u}) = \int_\Omega \operatorname{div} \boldsymbol{u} \, \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_{\partial\Omega} \varphi \cdot \boldsymbol{n} \, \mathrm{d}S = 0$$

$$(+) \;\Rightarrow\; \operatorname{div} \boldsymbol{u} = 0 \text{ — clear.}$$

अ Discrete problem

For simplicity we assume that $N = 2$, Ω is a polygonal domain, $\mathcal{T}_h$ is a tri- angulation of Ω with standard properties. This means that $K \in \mathcal{T}_h$ are closed triangles,

$$\overline{\Omega} = \cup K \in \mathcal{T}_h K$$

Over T_h we construct finite dimensional spaces and consider approximations

$$\begin{aligned}
&X_h \approx H^1(\Omega), \quad X_{h0} \approx H_0^1(\Omega), \quad \boldsymbol{V}_h \approx \boldsymbol{V}, \\
&\boldsymbol{X}_h \approx \boldsymbol{H}^1(\Omega), \quad \boldsymbol{X}_{h0} \approx \boldsymbol{H}_0^1(\Omega) \\
&M_h \approx L^2(\Omega), \quad M_{h0} \approx L_0^2(\Omega), \\
&\boldsymbol{X}_h, \boldsymbol{V}_h, \ldots \subset \boldsymbol{L}^2(\Omega), \; M_h \subset L^2(\Omega), M_{h0} \subset L_0^2(\Omega), \\
&((\cdot,\cdot))_h \approx ((\cdot,\cdot)), \quad |||\cdot|||_h \approx |||\cdot|||, \quad \boldsymbol{g}_h \approx \boldsymbol{g}, \quad \boldsymbol{u}_h^* \approx \boldsymbol{u}^*, \\
&\operatorname{div}_h \approx \operatorname{div}, b_h(\cdot,\cdot,\cdot) \approx b(\cdot,\cdot,\cdot)
\end{aligned}$$

CONCLUSION : Fluid-mechanics is an "ancient science" that is incredibly alive today. The modern technologies require a deeper understanding of the behavior of real fluids; on the other hand new discoveries often pose new challenging mathematical problems.The basic characteristic of the fluid is that it can flow. Fluids are divided in two categories, incompressible fluids (fluids that move at far subsonic speeds and do not change their density) and compressible fluids The aim of this paper is to furnish some results in very different areas that are linked by the common scope of giving new insight in the field of fluid dynamics. The aim of this paper is to furnish some results in very different areas that are linked by the common scope of giving new insight in the field of fluid dynamics. The study of these flows has been attached with a wide range of mathematical techniques and, today, this is a stimulating part of both pure an applied mathematics.

REFERENCES :

1. Feistauer, M., Dolejˇs´ı, V., and Kuˇcera, V. (2005). On the discontinuous Galerkin method for the simulation of compressible flow with wide range of Mach numbers. The Preprint Series of the School of Mathematics MATH-KNM-2005/5Charles University, Prague.
2. Feistauer, M. and Felcman, J. (1997). Theory and applications of numericaschemes for nonlinear convection–diffusion problems and compressible viscous flow. In The Mathematics of Finite Elements and Applications, Highlight1996 (ed. J. Whiteman), pp. 175–194. Wiley, Chichester.
3. Franca, L. P., Hughes, T., Mallet, M., and Misukami, A. (1986). A new finite element formulation fluid dynamics, I. Comput. Methods Appl. Mech. Eng., 54, 223–234.

4. Lions, P. L. (1993). Existence globale de solutions pour les ´equations de Navier– Stokes compressibles isentropiques. C.R. Acad. Sci. Paris, 316, 1335–1340.
5. Lions, P. L. (1998). Mathematical Topics in Fluid Mechanics, Volume 2: Compressible Models. Oxford Science Publications, Oxford.
6. Lube, G. and Weiss, D. (1995). Stabilized finite element methods for singularly perturbed parabolic problems. Appl. Numer. Math., 17, 431–459.
7. Matsumura, A. and Padula, M. (1992). Stability of stationary flows of compressible fluids subject to large external potential forces. Stabil. Appl. Anal. Contin. Media, 2, 183–202.
8. Novotn´y, A. and Straˇskraba, I. (2003). Mathematical Theory of Compressible Flow. Oxford University Press, Oxford.
9. Oden, J. T., Babuˇska, I., and Baumann, C. E. (1998). A discontinuous hp finite element method for diffusion problems. J. Comput. Phys, 146, 491–519.
10. Y. Achdou,F. Nataf,P. Le Tallec and M. Vidrascu,Adomain decomposition preconditioner for an advection diffusion problem Tech. Report,1998.
11. R.A. Adams, Sobolev spaces,Academic Press,1975,Pure and Applied Mathematics, Vol. 65.
12. V.I. Agoshkov and V.I. Lebedev, Poincar´e-Steklov operators and methods of partition of the domain in variational problems,Computational processes and systems,No. 2 (Moscow),"Nauka",Moscow,1985,pp. 173–227.
13. L.C. Berselli, ANeumann-Neumann preconditioner for low-frequency time-harmonic Maxwell equations,Tech. Report 2.327.1158,Dept. of Mathematics,Pisa University, 1999,see also Quarteroni and Valli: Domain decomposition methods for Partial Differential Equations. Oxford Science Publications (1999),pp. 130-132.
14. L.C. Berselli, Sufficient conditions for the regularity of the solutions of the NavierStokes equations,Math. Meth. Appl. Sci. 22 (1999),1079–1085.