Fourier Series Analysis

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Abstract

Based on a Fourier series analysis, an analytic interconnect model is presented which is suitable for periodic signals, such as a clock signal. In this model, the far end time domain waveform is approximated by the summation of several sinusoids. Closed form solutions of the 50% delay are provided when the fifth and higher harmonics are ignored. The model is applied to distributed interconnect trees and multiple coupled interconnects. Good accuracy is observed between the model and SPICE simulations. The computational complexity of the model is linear with the number of harmonics.

Keywords

Fourier series; Math; Mathematics; Fourier Analysis

I. INTRODUCTION

In deep sub-micrometer integrated circuits, interconnect delay dominates the gate delay. Furthermore, wire inductances can no longer be ignored due to higher signal frequencies and longer wire lengths. Accurate and efficient RLC interconnect models are therefore critical in the design of high performance integrated circuits. Based on modified Bessel functions, expressions characterizing the transient response of an RLC interconnect have been rigorously developed in [1]. These results, however, are highly complicated and not suitable for an exploratory design process. In order to produce a more efficient solution, the transfer function of the interconnect is truncated and approximated with a few dominant poles, for example, two poles in [2], and four poles in [3]. Four pole expressions are highly accurate, however, no closed form solution has been developed [3]. Furthermore, on-chip interconnect often has complicated structures, such as distributed RLC trees and buses. Interconnect models should have the ability to characterize these types of structures. Fourier analysis has been widely used in RF circuit simulation, where it is named harmonic balance [4]. In this paper, Fourier series analysis is applied to digital integrated circuits to model the interconnect behavior. The model is suitable for periodic signals, such as a clock signal. Since the solution is the steady state response, initial conditions are considered.

The basic idea of the Fourier series is that any periodic waveform can be represented with a sum of harmonically related sinusoids. Let's break this statement down. First, a waveform is a function of time, such as the one shown in Figure 1. A waveform is periodic if it repeats itself identically after a period of time. Let the period be denoted T. Then mathematically, a T-periodic waveform v satisfies a periodic waveform with period T (2) for all t. To make things simpler, let's further assume that v is a continuous function of time.

II. SINGLE INTERCONNECT MODEL

A classical interconnect model. The interconnection is represented by a distributed RLC transmission line, where l is the interconnect length, and R, L, C are the resistance, inductance, and capacitance per unit length, respectively. The driver is linearized as a voltage source Vin serially connected with a driver resistance Rd. The load of the interconnection is modeled as a capacitor Cl. From the ABCD parameters [5] of a transmission line, the transfer function from Vin to the far end of a line is

H(s)= 1 (1 + RdCls)coshθ +(Rd/Zc + ZcCls)sinhθ, (1)

where θ = l(R + sL)sC and Zc = (R + sL)/sC. Since (1) includes hyperbolic functions of the complex frequency s, it is difficult to obtain the time domain solution through an inverse Laplace transform. In order to simplify the problem, the denominator of the transfer function is expanded into an infinite series. By truncating this series, the transfer function is approximated by a few dominant poles [2], [3]. A distributed RLC line can also be modeled by lumped elements through moment matching [6].

In Fig. 2, the transfer function of some existing models [2], [3], [6] are compared with the exact transfer function described in (1).

III. APPLICATION EXAMPLES

The solution for a single distributed RLC line can be readily extended to interconnect trees and multiple coupled interconnects.

A. Distributed interconnect trees Interconnect trees are widely used in digital integrated circuits, such as clock distribution networks. An example of a distributed RLC tree is shown in Fig. 4, where lx and Cx are the normalized reference length and capacitance, respectively. All of the branches in the tree are represented by distributed RLC lines.

Where θ and Zc are defined in section II. When a node has multiple fanout, the load impedance seen at this node is the parallel combination of the input impedance of the downstream branches. The transfer function from N0 to a certain node Ni is the product of the transfer function of all of the branches along the unique path from N0 to Ni. The transfer function of a single branch can be obtained by replacing Rd by 0 and Cls by 1/ZL in (1),

H(s)= 1 coshθ +(Zc/ZL)sinhθ. (14)

The methods presented in [2] and [9] have similar accuracy and complexity, since both of these models are based on second order approximations. As listed in Table I, Fb3 and Fb5 produce higher accuracy, for this example, than the second order approximations. The average error of Fb5 is only 3%. The accuracy of the Fourier series based model can be enhanced to capture the fine details of the waveform by including additional harmonics, and there are no stability and numerical problems such as suffered by AWE [10].

B. Multiconductor Systems

For multiple transmission lines, the interconnect parameters per unit length can be represented by matrices R, L, and C. All of these matrices are symmetric with the dimension N ×N, where N is the number of lines.

In general, M is a matrix function of s and cannot be expressed in closed form [11]. Furthermore, the matrix inverse operation in (20) does not permit an analytic expression (or an analytic low order approximation) of the transfer function to be obtained. Conventional inverse Laplace transform based methods [1]–[3], which assume a step or ramp input, can no longer be used. The proposed model, which assumes a periodic input signal, remains valid, since the solution of (20) is only required at certain discrete frequencies (e.g., the harmonic frequencies of the input signal), and can be solved numerically at each frequency. When N is less than five, closed form solutions exist [13] to calculate M and Q. For larger N, numerical methods have to be used, and the computing complexity increases. When s =0 , H becomes an identity matrix. Since no approximation is made in this derivation, (20) is the exact transfer function of a coupled multi- conductor system. Ground lines are placed on each side of the signal lines to provide current return paths.

IV. Exponential Form of the Fourier series

The Fourier series given in (4) is referred to as the trigonometric form. It is also referred to as the single-sided form because the Fourier coefficients all have non-negative indices (they are all on one side of zero). An alternate, often simpler form is the exponential form, also known as the double-sided form because the Fourier coefficients have both positive and negative indices. The exponential form uses complex numbers and is notationally simpler because you can use one complex coefficient to play the role of the two coefficients required per harmonic in the trigonometric form.

The exponential form of the Fourier series uses Euler's formula, jk t where j =–1 . Now, a more general Fourier series is

e0 = coski0t + jsinki0 (53)i jki0t.(54) viti = cke k = –i

This is more general in that it allows v to be complex, which is often not needed. Then v will be a real function. What this means is that ck must be the complex conjugate of c–k, or that the real parts of ck and c–k must be the same, but the imaginary parts must have opposite signs.

V. The Fourier Coefficients

The signal consists of three components, a DC component and two components at the fundamental frequency (cosine and sine). This is shown in Figure 3. When computing a0 the DC component is extracted from the composite signal by computing the average over exactly one period. The other components of the signal are at the fundamental frequency and so would be ignored because we integrate over one full cycle of these components and they are symmetric about zero over one period and so average to zero.

VI. Orthogonal Decomposition

In the above example you will see that for each coefficient, if we wanted ak, the coefficient of the kth harmonic cosine, we multiplied the signal by cos 2 kf0t , which translates the component of interest to DC, where it is extracted while discarding all other terms by integrating over exactly one cycle of the fundamental frequency. Similarly, if we are interested in bk, the coefficient of the kth harmonic sine, we multiply the signal by sin 2 kf0t , which translates the component of interest to DC, where it is extracted while discarding all other terms by integrating over exactly one cycle of the fundamental frequency.

VII. Properties of the Fourier Series

Many useful properties of the Fourier series are presented in this section and summarized in Table 1 on page 22.

Let F(x) denote a transformation of a T-periodic waveform x into its sequence of Fourier coefficients X by repeated application of (62), F(x) = X .(82)

F is a linear transformation and so superposition holds. In other words, assume that a and b are simple real numbers, that x and y are T- periodic functions, and that

Z(t) = a.x(t)+ b.y(t)

If F (x) = X , F (y) = Y , and F (z) = Z

A. Compact Groups

One of the interesting properties of the Fourier transform which we have mentioned, is that it carries convolutions to pointwise products. If that is the property which we seek to preserve, one can produce Fourier series on any compact group. Typical examples include those classical groups that are compact. This generalizes the Fourier transform to all spaces of the form L2(G), where G is a compact group, in such a way that the Fourier transform carries convolutions to pointwise products. The Fourier series exists and converges in similar ways to the [−π, π] case.

B. Riemannian manifolds

If the domain is not a group, then there is no intrinsically defined convolution. However, if X is a compact Riemannian manifold, it has a Laplace–Beltrami operator. The Laplace–Beltrami operator is the differential operator that corresponds to Laplace operator for the Riemannian manifold X. Then, by analogy, one can consider heat equations on X. Since Fourier arrived at his basis by attempting to solve the heat equation, the natural generalization is to use the eigensolutions of the Laplace–Beltrami operator as a basis. This generalizes Fourier series to spaces of the type L2(X), where X is a Riemannian manifold. The Fourier series converges in ways similar to the [−π, π] case.

C. Locally compact Abelian groups

The generalization to compact groups discussed above does not generalize to noncompact, nonabelian groups. However, there is a straightfoward generalization to Locally Compact Abelian (LCA) groups.

This generalizes the Fourier transform to L1(G) or L2(G), where G is an LCA group. If G is compact, one also obtains a Fourier series, which converges similarly to the [−π, π] case, but if G is noncompact, one obtains instead a Fourier integral. This generalization yields the usual Fourier transform when the underlying locally compact Abelian group is R.

Convergence

Because of the least squares property, and because of the completeness of the Fourier basis, we obtain an elementary convergence result.

Theorem. If f belongs to L2([−π, π]), then f∞ converges to f in L2([−π, π]), that is, $\|f_N - f\|_2$ converges to 0 as $N \to \infty$.

We have already mentioned that if f is continuously differentiable, then $(i\cdot n)\hat{f}(n)$ is the nth Fourier coefficient of the derivative f′. It follows, essentially from the Cauchy–Schwarz inequality, that f∞ is absolutely summable. The sum of this series is a continuous function, equal to f, since the Fourier series converges in the mean to f: Theorem. If $f \in C^1(\mathbb{T})$, then f∞ converges to f uniformly (and hence also pointwise.)

This result can be proven easily if f is further assumed to be C2, since in that case $n^2\hat{f}(n)$ tends to zero as $n \to \infty$. More generally, the Fourier series is absolutely summable, thus converges uniformly to f, provided that f satisfies a Hölder condition of order α > ½. In the absolutely summable case, the inequality $\sup_x |f(x) - f_N(x)| \le$

$\sum_{|n| > N} |\hat{f}(n)|$ proves uniform convergence.

Many other results concerning the convergence of Fourier series are known, ranging from the moderately simple result that the series converges at x if f is differentiable at x, to LennartCarleson's much more sophisticated result that the Fourier series of an L2 function actually converges almost everywhere.

These theorems, and informal variations of them that don't specify the convergence conditions, are sometimes referred to generically as "Fourier's theorem" or "the Fourier theorem"

CONCLUSIONS

By exploiting a Fourier series representation of a typical on-chip signal, an analytic time-domain solution for an RLC interconnect is

TABLE II COMPARISON OF THE MAXIMUM CROSSTALK NOISE OF Fb3 AND Fb5 WITH SPICE SIMULATIONS.THE INPUT SIGNAL PARAMETERS ARE T = 500ps, τ = 50ps , AND Vdd =1 .5volts. l Victim SPICE Fb3 Fb5 (mm) (mV) (mV) % Error (mV) % Error V2 155.9 131.4 15.7 151.9 2.6 2 V3 67.6 48.9 27.7 69.8 3.3 V4 54.6 39.3 28.0 57.5 5.3 V5 40.6 26.8 34.0 40.9 0.7 V2 190.5 197.0 3.4 195.2 2.5 4 V3 68.8 73.4 6.7 62.0 9.9 V4 60.3 54.4 9.8 54.0 10.4 V5 48.2 38.4 20.3 34.7 28.0 V2 188.8 201.8 6.9 192.4 1.9 6 V3 110.6 79.6 28.0 99.0 10.5 V4 95.0 66.7 29.8 87.4 8.0 V5 74.0 43.9 40.7 60.9 17.7

Shown to be an effective modeling strategy. Closed form solutions of the 50% delay are presented. The model is applied to distributed interconnect trees and multiple coupled interconnects, the transfer functions of which are exact. Good accuracy is observed between the proposed model and SPICE simulations.

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