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Existence of Cyclic Subgroup of order p of group of Automorphisms of a finite p-group

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Abstract: *We give definitions of some automorphisms and give some important results. We find condition that cyclic subgroup of order p of group of automorphisms of a finite p-group exists.*

Keywords: Inner automorphism, Central automorphism, outer automorphism

1. Introduction

Let G be a p-group. An isomorphism from a group G onto itself is called an automorphism of G. Aut(G) denote the set of all automorphisms of G. The automorphism f_a: G →G given by $f_a(x) = a^{-1}xa$ ∀ x ε G is called the inner automorphism of G induced by a. Inn(G) denotes the set of all inner automorphisms of G. The set of automorphisms of a group and the set of inner automorphisms of a group are both groups under the operation function composition [4]. Many problems are open for automorphisms of finite p-groups. There is conjecture of Berkovich stating that every non-simple finite p-group has a non-inner automorphism of order p [5]. This conjecture is confirmed for various classes of p-groups.

2,. Some definitions and results

2.1 [9] An automorphism φ of G is called a class-preserving automorphism, if for each x ε G, ∃ an element g_x ε G s.t. $\varphi(x) = g_x^{-1} x\, g_x$; and is called an inner automorphism of for each x ε G, ∃ a fixed element g ε G s.t. $\varphi(x) = g^{-1}xg$. An automorphism of a group is called an outer automorphism if it is not an inner automorphism. An automorphism φ of G is called a central automorphism If it commutes with all inner automorphisms or equivalently, if $g^{-1}\varphi(g)$ ε Z (G) ∀ g ε G.

The commutator of a, b ε G is $[a, b] = a^{-1}b^{-1}ab$ and the commutator subgroup G' of G is the subgroup of G generated by all commutators of G. A maximal subgroup of G is a proper subgroup S s.t. there is no subgroup H of G s.t. S ⊂ H ⊂ G. The Frattini subgroup φ(G) of G is the intersection of all maximal subgroups of G. If G has no maximal subgroup then φ(G) = G. An element a ε G is called non-generator of G, If G = <a, Y> then G = <Y>. φ(G) is exactly the set of all nongenerators of G. Let G be a finite group then G is nilpotent if and only of $G/\varphi(G)$ is nilpotent.

A cyclic group is generated by a single element. We denote a cyclic group of order n by C_n. The rank of a group is denoted by d(G), which is the smallest generating set of G. The least common multiple of orders of the elements of a finite group G is called the exponent of G.

[6] A finite collection of normal subgroups H_i of a group G is a normal series for G if

$1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots\ldots \subseteq H_r = G$. This normal series is a central series if $H_i/H_{i-1} \subseteq Z(G/H_{i-1})$ for $1 \leq i \leq r$. A group G is nilpotent if it has a central series. Subgroups and factor groups of nilpotent groups are nilpotent.

Given any group G, we define a central series as follows. Let $Z_0 = 1$ and $Z_1 = Z(G)$. The second center Z_2 is defined to be the unique subgroup such that $Z_2/Z_1 = Z(G/Z_1)$. We continue like this, inductively defining Z_n for $n > 0$ so that $Z_n/Z_{n-1} = Z(G/Z_{n-1})$.

The chain of normal subgroups.

$1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots\ldots$

Constructed in this way is called upper central series of G. The upper central series may not actually be a central series for G because it may happen that $Z_i < G$ for all i. If $Z_r = G$ for some integer r, then $\{Z_i \mid 0 \leq i \leq r\}$ is a true central series and G is nilpotent.

If G is an arbitrary nilpotent group, then G is a term of its upper central series. As $G = Z_r$ for some integer $r \geq 0$, and the smallest integer r for which this happens is called the nilpotence class of G. Non trivial abelian groups have nilpotence class 1, and for non-abelian groups of nilpotency class 2 have quotient group G/Z(G) abelian.

2.2 Lemma: ([6]) Let G be finite. Then the following are equivalent.

1. G is nilpotent
2. Every nontrivial homomorphic image of G has a nontrivial center.
3. G appears as a member of its upper central series.

2.3 Theorem. ([6]) Let G be a (not necessarily finite) nilpotent group with central series.

$1 = 1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots\ldots \subseteq H_r = G$,

and let $1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots\ldots$

be the upper central series of G. Then $H_i \subseteq Z_i$ for $0 \leq i \leq r$, and in particular, $Z_r = G$.

Theorem 2.4 ([6]) Let H is a subgroup of G, where G is a nilpotent group. Then $N_G(H) > H$.

Theorem 2.5 For any group G, Inn(G) is a normal subgroup of Aut(G). Further $Inn(G) \cong G/Z(G)$, where Z(G) denotes the centre of G.

3. Cyclic Subgroup of order p of

group of Automorphism of a finite p-group.

3.1 [10] First we find inner-automorphism of order p

Let G be a p-group If a ε Z(G), then clearly the inner automorphism f_a is trivial.

Now If a ∉Z(G) but o(a) = p, then first we prove that $f_a^p(x) = a^{-p} x a^p$

Let $f_a^2(x) = f_a(f_a(x)) = f_a(a^{-1} x a) = a^{-1}(a^{-1} x a) a = a^{-2} x a^2$

In the similar way we arrived at $f_a^p(x) = a^{-p} x a^p$, Now since o(a) = p ∴ $a^p = e$, e the identity of G, so $f_a^p(x) = x$

Since p is the smallest prime, therefore order of f_a is p.

Now Let a ∉ Z(G) and o(a) ≠ p. Since G/Z(G) is a p-group, therefore let order of aZ(G) is p. So $(aZ(G))^p = Z(G)$, $a^pZ(G) = Z(G)$. We see that a^p ε Z(G) and $f_a^p(x) = a^{-p} x a^p = x$

Because p is prime, so we conclude that If order of aZ(G) is p then f_a^p is identity. Therefore order of f_a is p.

Since f_a is an inner automorphism of order p. <f_a> is cyclic subgroup of Inn(G) of order p.

Further if G is non-abelian and we know that G/Z(G) ≅ Inn(G) If Inn(G) is cyclic which means G/Z(G) is cyclic. This contradicts the fact that G is non-abelian. Inn(G) is not cyclic for a non-abelian group G.

3.2 We know that the conjecture of Berkovich has been settled for p-group G [3] s.t.

(i) G is regular (ii) G' is cyclic (iii) G is nilpotent of class 2 or 3 (iv) G/Z(G) is powerful (v) G is of co-class 2.

Therefore we get a non-inner automophorism of order p. Let σ be the non-inner automorphism of order p.

Therefore σ generates the cyclic subgroup of order p consisting of non-inner automorphisms.

Theorem 3.3 Let G ≠ {e} be a cyclic group generated by b. Then any endomorphism φ of G is an automorphism of G If and only If f(b) is a generator of G. Further (i) If G is infinite then Aut(G) is of order 2. (ii) If G is of finite order n, then Aut(G) is isomorphic to the group of those positive integers ≤n, which are relatively prime to n, with binary operation as multiplication modulo n and order of Aut(G) is φ(n).

3.4 From the above theorem 3.3, Aut(G) is cyclic If G be a cyclic group.

3.5 The problem arises whether Aut(G) is cyclic for a non-cyclic group G.

Thus there is cyclic subgroup of order p of group of automorphisms. The result is open for discussion.

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