

# Application of Matrices

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## ABSTRACT

*This paper represent an approach to basic arithmetic between abstract matrices, i.e., matrices of symbolic dimension with underspecified components. We define a simple basis function that enables the representation of abstract matrices composed of arbitrary regions in a single term that supports matrix addition and multiplication by regular arithmetic on terms. This can, in particular, be exploited to obtain general arithmetic closure properties for classes of structured matrices. We also describe an approach using alternative basis functions that allow more compact expressions.*

## INTRODUCTION

Matrices find applications in almost every branch of science, engineering, economics, probability, theory, and statistics to mention a few. The aim of this section is to present the concept of matrix and their elementary properties. It is everyday mathematical practice to represent matrices in an abstract way with symbolic dimensions and containing underspecified parts described by the use of ellipses. While reasoning about matrices in this form is mathematically routine, there is very little automated support for it. In earlier work we have investigated the problem of representing abstract matrices

with certain entries given by expressions and others given by interpolating ellipses. Their analysis included determining conditions for boundaries between regions and general expressions for elements within regions of such matrices and has led to a representation that made abstract matrices available as a template for concrete matrices with fully specified dimensions and entries.

## DEFINITION

A matrix is a rectangular array of numbers or other mathematical objects, for which operations such as addition and multiplication are defined. Most commonly, a matrix over a field  $F$  is a rectangular array of scalars from  $F$ . Most of this article focuses on real and complex matrices, i.e., matrices whose elements are real numbers or complex numbers, respectively. More general types of entries are discussed below. For instance, this is a real matrix: The numbers, symbols or expressions in the matrix are called its entries or its elements. The horizontal and vertical lines of entries in a matrix are called rows and columns, respectively.

Example:

Consider the  $3 \times 4$  matrix represent respectively the 2-nd row and the 3-rd

column of the matrix, and 5 represents the entry in the matrix on the 2-nd row and 3-rd column.

$$A = \begin{vmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{vmatrix}$$

### SIZE

The size of a matrix is defined by the number of rows and columns that it contains. A matrix with  $m$  rows and  $n$  columns is called an  $m \times n$  matrix or  $m$ -by- $n$  matrix, while  $m$  and  $n$  are called its dimensions. For example, the matrix  $A$  above is a  $3 \times 4$  matrix. Matrices which have a single row are called row vectors, and those which have a single column are called column vectors. A matrix which has the same number of rows and columns is called a square matrix. A matrix with an infinite number of rows or columns (or both) is called an infinite matrix. In some contexts, such as computer algebra programs, it is useful to consider a matrix with no rows or no columns, called an empty matrix.

### MATRIX MULTIPLICATION

*Multiplication* of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If  $A$  is an  $m$ -by- $n$  matrix and  $B$  is an  $n$ -by- $p$  matrix, then

their *matrix product*  $AB$  is the  $m$ -by- $p$  matrix whose entries are given by dot product of the corresponding row of  $A$  and the corresponding column of  $B$ :

Matrix multiplication satisfies the rules  $(AB)C = A(BC)$  (associativity), and  $(A+B)C = AC+BC$  as well as  $C(A+B) = CA+CB$  (left and right distributive), whenever the size of the matrices is such that the various products are defined.[13] The product  $AB$  may be defined without  $BA$  being defined, namely if  $A$  and  $B$  are  $m$ -by- $n$  and  $n$ -by- $k$  matrices, respectively, and  $m \neq k$ . Even if both products are defined, they need not be equal, i.e., generally

$$AB \neq BA,$$

i.e., matrix multiplication is not commutative, in marked contrast to (rational, real, or complex) numbers whose product is independent of the order of the factors. An example of two matrices not commuting with each other is:

### ROW OPERATION

There are three types of row operations:

**Row addition:** that is adding a row to another.

**Row multiplication:** that is multiplying all entries of a row by a non-zero constant.

**Row switching:** that is interchanging two rows of a matrix.

These operations are used in a number of ways, including solving linear equations and finding matrix inverses.

## SUB MATRIX

A sub matrix of a matrix is obtained by deleting any collection of rows or columns.

### For example:

For any given 3-by-4 matrix, we can construct a 2-by-3 sub matrix by removing row 3 and column 2.

## LINEAR TRANSFORMATION

Matrices and matrix multiplication reveal their essential features when related to linear transformations, also known as linear maps. A real  $m$ -by- $n$  matrix  $A$  gives rise to a linear transformation  $R^n \rightarrow R^m$  mapping each vector  $x$  in  $R^n$  to the (matrix) product  $Ax$ , which is a vector in  $R^m$ . Conversely, each linear transformation  $f: R^n \rightarrow R^m$  arises from a unique  $m$ -by- $n$  matrix  $A$ : explicitly, the  $(i, j)$ -entry of  $A$  is the  $i$ th coordinate of  $f(e_j)$ , where  $e_j = (0, \dots, 0, 1, 0, \dots, 0)$  is the unit vector with 1 in the  $j$ th position and 0 elsewhere.

Furthermore, this fact indicates that the representation could be used as a starting point for translating the algebraic expression back into a graphical representation of the abstract matrix, i.e. ellipses etc. While our approach is encompassing enough to deal with abstract matrices of arbitrary structure we have identified cases in which the growth of terms is potentially exponential. We have started addressing this issue by investigating an alternative basis function that can avoid this exponential growth. Although the alternative basis function

requires distinct representations of abstract matrices, depending on the arithmetic operation we want to perform, we could already show that it is effective for addition and multiplication on abstract vectors. However, its correct extension to the two-dimensional case of abstract matrices remains the subject of future.

## DETERMINANT

The determinant  $\det(A)$  or  $|A|$  of a square matrix  $A$  is a number encoding certain properties of the matrix. A matrix is invertible if and only if its determinant is nonzero. Its absolute value equals the area (in  $R^2$ ) or volume (in  $R^3$ ) of the image of the unit square (or cube), while its sign corresponds to the orientation of the corresponding linear map: the determinant is positive if and only if the orientation is preserved. The determinant of 2-by-2 matrices is given by  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ . The determinant of 3-by-3 matrices involves 6 terms. The more lengthy Leibniz formula generalises these two formulae to all dimensions. The determinant of a product of square matrices equals the product of their determinants:  $\det(AB) = \det(A) \cdot \det(B)$ . Adding a multiple of any row to another row, or a multiple of any column to another column, does not change the determinant. Interchanging two rows or two columns affects the determinant by multiplying it by  $-1$ . Using these operations, any matrix can be transformed to a lower (or upper) triangular matrix, and for such matrices the determinant equals the product of the entries on the main diagonal; this provides a method to calculate the determinant of any matrix.

Finally, the Laplace expansion expresses the determinant in terms of minors, i.e., determinants of smaller matrices. This expansion can be used for a recursive definition of determinants (taking as starting case the determinant of a 1-by-1 matrix, which is its unique entry, or even the determinant of a 0-by-0 matrix, which is 1), that can be seen to be equivalent to the Leibniz formula. Determinants can be used to solve linear systems using Cramer's rule, where the division of the determinants of two related square matrices equates to the value of each of the system's variables. Eigen values and eigen vectors A number  $\lambda$  and a non-zero vector  $v$  satisfying

$$Av = \lambda v$$

are called an eigen value and an eigenvector of  $A$ , respectively.[nb 1][31] The number  $\lambda$  is an eigen value of an  $n \times n$ -matrix  $A$  if and only if  $A - \lambda I_n$  is not invertible, which is equivalent to  $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ . The polynomial  $p_A$  in an indeterminate  $X$  given by evaluation the determinant  $\det(XI_n - A)$  is called the characteristic polynomial of  $A$ . It is a monic polynomial of degree  $n$ . Therefore the polynomial equation  $p_A(\lambda) = 0$  has at most  $n$  different solutions, i.e., eigen values of the matrix. They may be complex even if the entries of  $A$  are real. According to the Cayley-Hamilton theorem,  $p_A(A) = 0$ , that is, the result of substituting the matrix itself into its own characteristic polynomial yields the zero matrix.

## SQUARE MATRIX

A square matrix is a matrix with the same number of rows and columns. An  $n$ -by- $n$

matrix is known as a square matrix of order  $n$ . Any two square matrices of the same order can be added and multiplied. The entries  $a_{ii}$  form the main diagonal of a square matrix. They lie on the imaginary line which runs from the top left corner to the bottom right corner of the matrix

## SYMMETRIC AND SKEW SYMMETRIC

A square matrix  $A$  that is equal to its transpose, i.e.,  $A = A^T$ , is a symmetric matrix. If instead,  $A$  was equal to the negative of its transpose, i.e.,  $A = -A^T$ , then  $A$  is a skew-symmetric matrix. In complex matrices, symmetry is often replaced by the concept of Hermitian matrices, which satisfy  $A^* = A$ , where the star or asterisk denotes the conjugate transpose of the matrix, i.e., the transpose of the complex conjugate of  $A$  by the spectral theorem, real symmetric matrices and complex Hermitian matrices have an eigen basis, i.e., every vector is expressible as a linear combination of eigenvectors. In both cases, all eigen values are real. This theorem can be generalized to infinite-dimensional situations related to matrices with infinitely many rows and columns, see below.

## DEFINITE MATRIX

A symmetric  $n \times n$ -matrix is called positive-definite (respectively negative-definite; indefinite), if for all nonzero vectors  $x \in \mathbb{R}^n$  the associated quadratic form given by  $Q(x) = x^T A x$  takes only positive values (respectively only negative values; both some negative and some positive values). If the quadratic form takes only non-negative (respectively only non-positive)

values, the symmetric matrix is called positive-semi-definite (respectively negative-semi-definite); hence the matrix is indefinite precisely when it is neither positive-semi-definite nor negative-semi-definite. A symmetric matrix is positive-definite if and only if all its eigen values are positive, i.e., the matrix is positive-semi-definite and it invertible.[24] The table at the right shows two possibilities for 2-by-2 matrices. Allowing as input two different vectors instead yields the bilinear form associated to A.

### Orthogonal matrix:

An orthogonal matrix is a square matrix with real entries whose columns and rows are orthogonal unit vectors. Equivalently, a matrix A is orthogonal if its transpose is equal to its inverse  $A^{\mathrm{T}}=A^{-1}$ , which entails  $A^{\mathrm{T}} A = A A^{\mathrm{T}} = I$ , where I is the identity matrix.

An orthogonal matrix A is necessarily invertible (with inverse  $A^{-1} = A^{\mathrm{T}}$ ), unitary ( $A^{-1} = A^*$ ), and normal ( $A^* A = A A^*$ ). The determinant of any orthogonal matrix is either +1 or -1. A special orthogonal matrix is an orthogonal matrix with determinant +1. As a linear transformation, every orthogonal matrix with determinant +1 is a pure rotation, while every orthogonal matrix with determinant -1 is either a pure reflection, or a composition of reflection and rotation. The complex analogue of an orthogonal matrix is a unitary matrix have not yet successfully extended them to full matrices.

### CONCLUSION

We have presented a computational approach to arithmetic on abstract matrices. We have defined a simple basis function that allows us to represent every abstract matrix regardless of its structural composition as a sum of region terms. Given this representation we could define matrix addition and multiplication straightforwardly as addition and multiplication of the sums. In fact we could show that presentation enables symbolic computations on abstract matrices that are considered mathematically routine but for which only limited automated support exists. In a next step we therefore intend to implement bespoke algorithms for abstract matrix arithmetic and combine them with our parsing procedure presented in [5]. Moreover, we intend to use our representation as a basis for developing other operations on abstract matrices such as computing Jordan normal forms or determinants. Another advantage of our representation is that the result of an arithmetic operation on two abstract matrices can be examined by systematic arithmetic manipulations and exploitation of the partial order structure of the basis function to yield structural properties of the resulting matrix. This could be further exploited to perform and automate general proofs of closure properties for certain classes of structural matrices

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