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New Laplace, *z* and Fourier-related transforms

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Abstract

In this paper, the author uses his recently proposed complex variable generalized distribution theory to expand the domains of existence of bilateral Laplace and z transforms, as well as a whole new class of related transforms. A vast expansion of the domains of existence of bilateral Laplace and z transforms and continuous-time and discrete-time Hilbert, Hartley and Mellin transforms, as well as transforms of multidimensional functions and sequences are obtained. It is noted that the Fourier transform and its applications have advanced by leaps and bounds during the last century, thanks to the introduction of the theory of distributions and, in particular, the concept of the Dirac-delta impulse. Meanwhile, however, the truly two-sided 'bilateral' Laplace and z transforms, which are more general than Fourier, remained at a standstill incapable of transforming the most basic of functions. In fact, they were reduced by half to one-sided transforms and received no more than a passing reference in the literature. It is shown that the newly proposed generalized distributions expand the domains of existence and application of Laplace and z transforms similar to and even more extensively than the expansion of the domain of Fourier transform that resulted from the introduction, nearly a century ago, of the theory of distributions and the Dirac- delta impulse. It is also shown that the new generalized distributions put an end to an anomaly that still exists today, which meant that for a large class of basic functions, the Fourier transform exists while the more general Laplace and z transforms do not. The anomaly further manifests itself in the fact that even for the one-sided causal functions, such as the Heaviside unit step function u(t) and the sinusoid sin βtu(t), the Laplace transform does not exist on the jω-axis, and the Fourier transform which does exist cannot be deduced thereof by the substitution s=jω in the Laplace transform, which by definition it should. The extended generalized transforms are well defined for a large class of functions ranging from the most basic to highly complex fast-rising exponential ones that have so far had no transform. Among basic applications, the solution of partial differential equations using the extended generalized transforms is provided. This paper clearly presents and articulates the significant impact of extending the domains of Laplace and z transforms on a large family of related transforms, after nearly a century during which bilateral Laplace and z transforms of even the most basic of functions were undefined, and the domains of definition of related transforms such as Hilbert, Hartley and Mellin transforms were confined to a fraction of the space they can now occupy.

Keywords:
Complex variables; Laplace and z transform; Fourier transform; Dirac-delta impulse

1. Introduction

Introduced towards the end of the nineteenth century, the theory of generalized functions, also known as the theory of distributions, has presented a powerful tool for dealing with pseudo-functions including impulsive phenomena, absolutely divergent integrals and their Fourier transforms. The effect of the introduction of distributions such as the Dirac-delta function on the evolution and importance of Fourier transform was dramatic, enlarging its domain of convergence to a large class of functions that had no transform. The same advances were subsequently made in the domain of discrete-time functions and sequences. Since the Dirac-delta impulse and its family of derivatives were defined on the line and not on the complex plane, the result was that while, for more than a century, the Fourier transform and its applications advanced and expanded by leaps and bounds, the more general 'bilateral' Laplace and *z* transforms stayed frozen in time, incapable of transforming the most basic of functions. In fact, as will be observed, the absence of a generalization of the Dirac-delta impulse has led to the present condition that some elementary functions such as unity or a two-sided infinite duration pure sinusoid have a Fourier transform but no Laplace or *z* transform. The condition is unusual since the Fourier transform is but a special case of the more general Laplace and *z* transforms.

Recently, a generalization of the Dirac-delta impulse (Corinthios 2003) and an extension of distribution theory to generalized functions of a complex variable (Corinthios 2005) were proposed. The objective of this paper is to apply the proposed new generalized distributions to extend further the domains of existence of Laplace and *z* transforms as well as generalizing an important class of related transforms, including Hilbert, Hartley and Mellin transforms. The new extended domains cover a large class of fast-rising functions that have hitherto had no transforms. Other applications of the generalized extended transforms are provided, including the solution of partial differential equations.

2. The absence of Laplace and *z* transforms of basic functions

The absence of Laplace and *z* transforms of elementary functions which have a Fourier transform is revealed by noticing that functions as simple as $f(t)=1$ and $g(t)=\cos\beta t$, and their discrete-time counterpart, have the Fourier transforms $F(\mathrm{j}\omega)=2\pi\delta(\omega)$ and $G(\mathrm{j}\omega)=\pi\{\delta(\omega-\beta)+\delta(\omega+\beta)\}$, respectively, but have neither Laplace nor *z* transform (e.g. Lepage 1961; Papoulis 1962; Abramowitz & Stegun 1964; Doetsch 1974; Kreyzig 1979; Davies 1985; Fogiel *et al.* 1986; Oppenheim & Schafer 1998; Gradshteyn & Ryzhik 2000; Poularikas 2000, p. 331). In fact, the whole class of functions that have impulsive spectra have neither Laplace nor *z* transform.

In the literature, the bilateral Laplace and *z* transforms of two-sided infinite duration functions is evaluated as the sum of the transform of a right-sided part and that of a left-sided part. The transforms of the right part and that of the left part are, however, mutually exclusive so that transforms of basic functions such as 1, $e^{\alpha t}$, $\cos\beta t$, $te^{\alpha t}$, $\cos\beta t$ and their discrete counterparts do not exist.

The discrepancy between Laplace and z transforms on the one hand, and Fourier transform on the other, also manifests itself in the fact that even for causal functions, such as the Heaviside unit step function $u(t)$ and the causal sinusoid $\sin\beta t\ u(t)$, the Laplace transform does not converge on the $j\omega$-axis implying the non-existence of the Fourier transform, and the substitution $s=j\omega$ in the Laplace transform does not produce the Fourier transform, a contradiction with the transforms' definitions.

The absence of a generalization of the theory of distributions and the Dirac-delta impulse for continuous-time functions, as well as discrete-time functions, has meant that neither Fourier, Laplace nor z transform tables in the mathematics handbooks and well-established references, such as those cited above, show transforms of two-sided infinite duration functions such as e^t, $e^t\sin\beta t$ or $te^t\sin\beta t$ or their discrete-time versions.

3. Distributions and their properties

The origins of the theory of distributions date back to works in the 1800s by Oliver Heaviside (1850–1925). During the last century, the theory of generalized functions was extensively developed in the works by Paul Adrien Maurice Dirac (1902–1984), Sergei Lvovich Sobolev (1908–1989), Laurent Schwartz (1915–2002) and Israil Moiseevic Gelfand (1913–).

A distribution or generalized function has properties that are foreign to usual functions. It differs from an ordinary function in the fact that whereas a function $f(t)$ is defined for all values of the independent variable t, a distribution $g(t)$ is not. The value of a distribution $g(t)$ is given by its inner product with a 'test function' $\phi(t)$. Test functions are infinitely differentiable and decay more rapidly than any power of t as $t\rightarrow\pm\infty$.

A distribution $g(t)$ is in fact a mapping that associates with any test function $\phi(t)$, a number which we may call $N_g[\phi(t)]$. This number is the inner product, also called 'scalar product' of the distribution $g(t)$ with the test function $\phi(t)$. We write $N_g[\phi(t)] = \langle g(t), \phi(t)\rangle = \int_{-\infty}^{\infty} g(t)\phi(t)\mathrm{d}t$, assuming that the integral exists. The impulse function distribution $\delta(t)$ is defined by stating that $N_\delta[\phi(t)] = \langle \delta(t), \phi(t)\rangle = \int_{-\infty}^{\infty} \delta(t)\phi(t)\mathrm{d}t = \phi(0).$ This definition of the delta function stands for a limit of a sequence of inner products, namely

$$\langle \delta_n(t), \phi(t)\rangle = \int_{-\infty}^{\infty} \delta_n(t)\phi(t)\mathrm{d}t \xrightarrow[n\rightarrow\infty]{} \int_{-\infty}^{\infty} \delta(t)\phi(t)\mathrm{d}t = \langle \delta(t), \phi(t)\rangle = \phi(0),$$

where $\delta_n(t)$ is a sequence of ordinary functions. Symbolically, this relation takes the form $\delta_n(t) \xrightarrow[n\rightarrow\infty]{} \delta(t)$, meaning that the sequence $\delta_n(t)$ tends in the limit to the delta distribution $\delta(t)$.

The inner product $\langle\delta(t), \phi(t)\rangle = \phi(0)$ may be adopted as the definition of the delta distribution, without reference to an integral that is in fact not well defined. For the distribution $\delta(t)$, the same property applies for any function $f(t)$, with the only condition that it be continuous at $t=0$. In this case, the same property takes the form $\int_{-\infty}^{\infty} \delta(t)f(t)\mathrm{d}t = f(0).$ Properties of distributions are well covered in the literature (Schwartz 1950; Lighthill 1959; Gelfand & Shilov 1964; Zemanian 1965; Bracewell 2000). In what follows, we study an overview of the newly proposed generalized distributions and, in particular, the generalized Dirac-delta impulse and its family of derivatives.

4. Generalized distributions for Laplace domain

We may define a generalized distribution $G(s)$, associated with Laplace transform complex domain, as a generalized function of a complex variable $s=\sigma+j\omega$, which may be defined as an integral along a straight-line contour in the s-plane extending from a point $s=\sigma-j\infty$ to $\sigma+j\infty$ of the product of $G(s)$ with a test function $\Phi(s)$. For convenience, we refer to this integral by the symbol $I_G[\Phi(s)]$, or simply $I_G[\Phi]$, and use the shorthand notation

$$I_G[\Phi(s)] = \langle G(s), \Phi(s)\rangle_{\Re[s]=\sigma} = \int_{\sigma-j\infty}^{\sigma+j\infty} G(s)\Phi(s)\mathrm{d}s.$$

The test function $\Phi(s)$ has derivatives of any order along straight lines in the s-plane going through the origin, and approach zero more rapidly than any power of $|s|$. For example, if the generalized distribution is the generalized impulse $\xi(s)$ (Corinthios 2005), we may write

$$I_G[\Phi(s)] = \langle \xi(s), \Phi(s)\rangle_{\Re[s]=\sigma} = \int_{\sigma-j\infty}^{\sigma+j\infty} \xi(s)\Phi(s)\mathrm{d}s = \begin{cases} j\Phi(0), & \sigma=0, \\ 0, & \sigma\neq 0. \end{cases}$$

A three-dimensional generalization of the time domain test function may be written as the function shown in figure 1,

$$\Phi(s) = \begin{cases} e^{-1/(|s|^2-1)}, & |s|<1, \\ 0, & |s|\geq 1. \end{cases}$$

The properties of the generalized distributions are generalizations of those of the well-known properties of the theory of distributions (Corinthios 2005). In what follows, we focus our attention on the generalized Dirac-delta impulse and its application to transforms of general two-sided and one-sided functions and sequences.

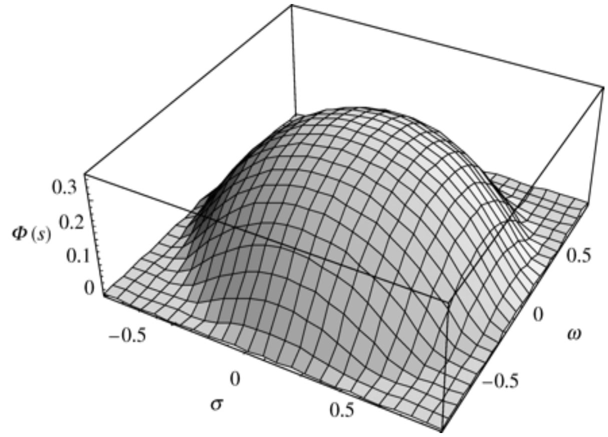

Figure 1

A three-dimensional generalized test function $\Phi(s)$.

5. Complex s-domain generalization of the Dirac-delta impulse

The generalized Dirac-delta impulse denoted as $\xi(s)$ was introduced by Corinthios (2003) using a definition based on its integral about the origin. The generalized complex domain distributions lead to a more suitable definition in terms of complex test functions. The following properties of the generalized Dirac-delta impulse are easily established:

$$\langle \xi(s), \Phi(s)\rangle_{\Re[s]=\sigma} = \begin{cases} \int_{-j\infty}^{j\infty} \xi(s)\Phi(s)\mathrm{d}s = \int_{-\infty}^{\infty} \delta(\omega)\Phi(j\omega)j\,\mathrm{d}\omega = j\Phi(0), & \sigma=0, \\ 0, & \sigma\neq 0. \end{cases}$$

If $F(s)$ is analytic at $s=0$, then

$$\langle \xi(s), F(s)\rangle_{\Re[s]=\sigma} = \begin{cases} \int_{-j\infty}^{j\infty} \xi(s)F(s)\mathrm{d}s = jF(0), & \sigma=0, \\ 0, & \sigma\neq 0. \end{cases}$$

The Dirac-delta impulse may be viewed as the limit of a sequence of functions, such as a rectangle, which progressively decrease in time and increase in height. In the complex s-plane, the process may be viewed as the three-dimensional solid

$X_\varepsilon(s) = \begin{cases} 1/\varepsilon, & |s| < \varepsilon/2, \\ 0, & |s| > \varepsilon/2, \end{cases}$ shown in figure 2, which tends to a generalized impulse as $\epsilon \to 0$. Another sequence leading to the Dirac-delta impulse is the Gaussian sequence $v(t) = \mathrm{e}^{-t^2/\varepsilon}/\sqrt{\pi\varepsilon}$, which can be viewed as the three-dimensional function on the s-plane $X_\varepsilon(s) = \mathrm{e}^{-|s|^2/\varepsilon}/\sqrt{\varepsilon\pi}$, shown in figure 3.

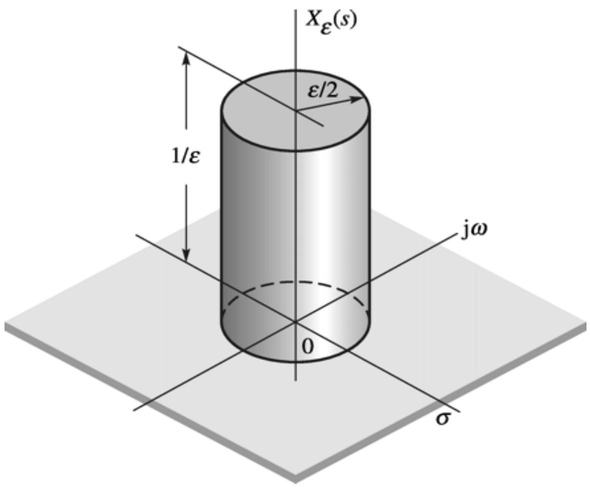

Figure 2

Cylinder as a three-dimensional object leading to the generalized impulse $\xi(s)$.

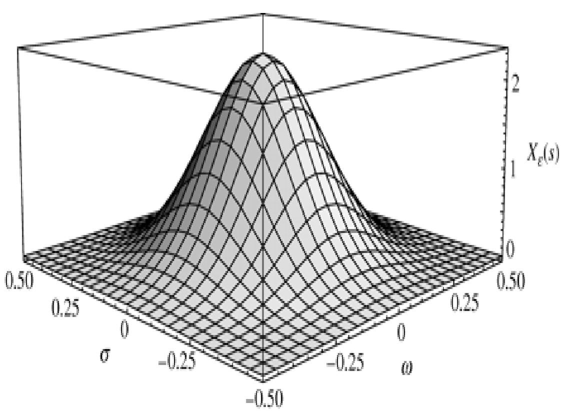

Figure 3

A three-dimensional generalization of the Gaussian function in s domain.

Using the complex s-domain generalized impulse, we can write the Laplace transform of unity, namely $\int_{-\infty}^{\infty} \mathrm{e}^{-st}\mathrm{d}t = 2\pi\xi(s)$, of which the Fourier transform of unity is but the special case $\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = 2\pi\delta(\omega)$, obtained by setting $\sigma=\mathrm{j}\omega$. This transform leads to a whole new class of bilateral Laplace transforms (Corinthios 2003). Some such transforms are listed in table 1.

6. *z*-transform domain

The properties of generalized distributions on the complex z-plane have been recently explored (Corinthios 2005). A distribution $G(z)$ may be defined as the value of the integral, denoted as $I_G[\Phi(z)]$, of its product with a test function $\Phi(z)$. Symbolically, we may write

$$I_G[\Phi(z)] = \langle G(z), \Phi(z)\rangle_{|z|=r} = \oint_{|z|=r} G(z)\Phi(z)\mathrm{d}z,$$

where the contour of integration is a circle of radius $r=|z|$ centred at the origin in the z-plane. Similar to the test function $\Phi(s)$ shown above, a possible generalization of a test function into a three-dimensional solid can be defined as the function $\Phi(z)$ given by

$$\Phi(z) = \begin{cases} \mathrm{e}^{-1/(|z-1|^2-1)}, & |z-1| < 1, \\ 0, & |z-1| \geq 1, \end{cases}$$

shown in figure 4.

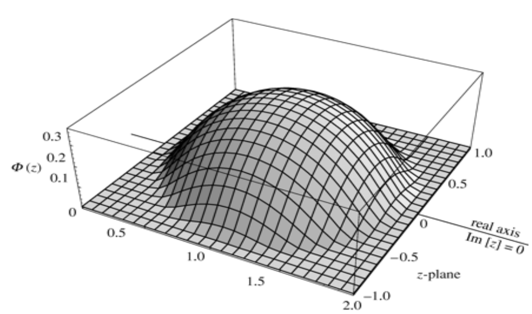

Figure 4

A three-dimensional test function $\Phi(z)$ in z domain.

When seen along the arc of the unit circle, the test function has the form $\Phi(e^{j\Omega})$ depicted in figure 5. It is noted that the function has the property of terminating at $\Omega=\pm1$ with all derivatives equal to zero. It also has the property that all the terms in its Maclaurin series are zero. Since the definition of the proposed generalized impulse involves integration along the unit circle, the resulting properties of the integration with a test function reduce to integration along a line as in the definition of the Dirac-delta impulse in the context of the usual distribution theory. It is to be noted, moreover, that in general new transforms can be evaluated using the basic definition and properties of the generalized impulse and its derivatives without the need to employ rigorous justification using test functions (Corinthios 2003).

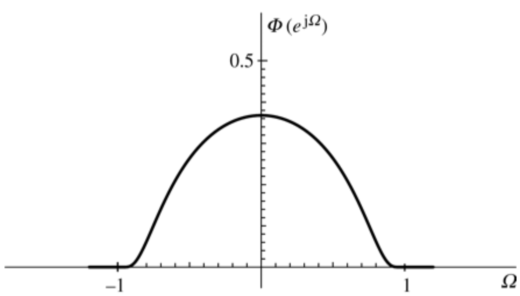

Figure 5

Test function in z-plane as seen along the arc of the unit circle.

7. Complex z-domain generalization of the Dirac-delta impulse

The discrete-time domain generalized impulse will be denoted by the symbol $\psi(z)$ and is equivalent to the symbol $\zeta(z-1)$ used earlier (Corinthios 2003), i.e.

$$\psi(z)=\zeta(z-1),$$

$$\langle\psi(z),\Phi(z)\rangle_{|z|=r}=\begin{cases} j\Phi(1), & r=1,\\ 0, & r\neq 1.\end{cases}$$

If $X(z)$ is analytic at $z=1$, then

$$\oint_{|z|=r}\psi(z)F(z)\mathrm{d}z=\begin{cases} jF(1), & r=1,\\ 0, & r\neq 1.\end{cases}$$

In the z-plane, the cylinder

$$X_\varepsilon(z)=\begin{cases} 1/\varepsilon, & |z-1|<\varepsilon/2,\\ 0, & |z-1|>\varepsilon/2,\end{cases}$$

shown in figure 6 may be viewed as a three-dimensional solid corresponding to the usual rectangular sequence leading to impulses.

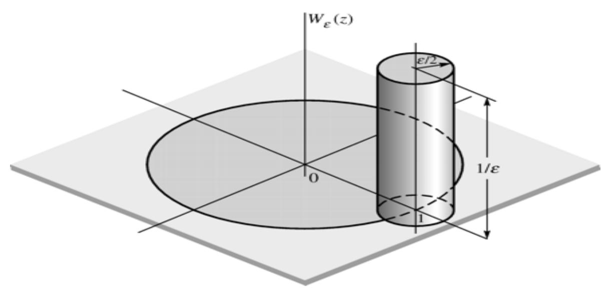

Figure 6

A cylindrical sequence on z-plane corresponding to the generalized impulse $\psi(z)$.

A possible three-dimensional solid generalization of the Gaussian sequence leading to the impulse may be defined as the sequence

$$W_\varepsilon(z)=\frac{1}{\sqrt{\pi\varepsilon}}e^{-|z-1|^2/\varepsilon},$$

shown in figure 7.

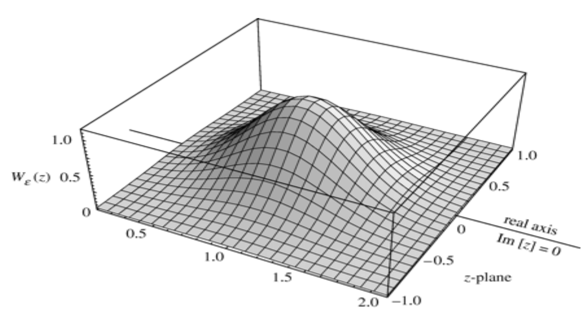

Figure 7

A three-dimensional Gaussian sequence leading to the impulse $\psi(z)$ in z domain.

Among the properties of the $\psi(z)$ generalized impulse is the convolution in the z domain given by $\psi(z/a) * \psi(z/b) = \mathrm{j}\psi[z/(ab)]$, an operation illustrated in figure 8.

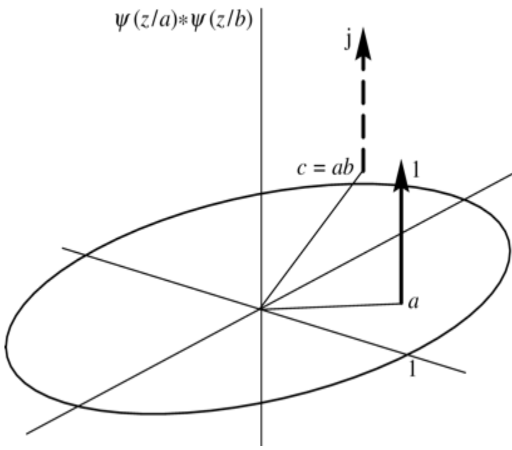

Figure 8

Convolution of impulses on z-plane.

The generalized z-domain impulse leads to the expansion of the domain of existence of the bilateral z transform (Corinthios 2005). some such new extended z transforms. In what follows, other Fourier-, Laplace- and z-related transforms are generalized by applying the extended Laplace and z transforms.

8. Hilbert transform generalization

The Hilbert transform of a function $f(t)$ may be defined as a transformation from the time domain to the time domain, producing a function $f_{\mathrm{Hi}}(t)$, such that $f_{\mathrm{Hi}}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\tau)}{t-\tau}\mathrm{d}\tau = f(t) * h(t)$, where $h(t)=1/(\pi t)$. Equivalently, we may consider the Hilbert transform as a transformation from the time domain to Fourier frequency domain $F_{\mathrm{Hi}}(\mathrm{j}\omega) = F[\mathrm{j}\omega]H[\mathrm{j}\omega]$, where $H(\mathrm{j}\omega) = \mathcal{F}\left[\frac{1}{\pi t}\right] = -\mathrm{j}\,\mathrm{sgn}(\omega)$. The generalization of the Hilbert transform would be a transformation from the time domain to the bilateral Laplace transform domain. We may write $G(s) = H(s)F(s)$, $H(s) = \mathcal{L}\left[\frac{1}{\pi t}\right] = -\mathrm{j}\,\mathrm{sgn}(-\mathrm{j}s)$. The signum function of a complex variable $\mathrm{sgn}(s)$ would have to be defined. We may write $\mathrm{sgn}(w) = \mathrm{sgn}(\Re[w])$, so that $\mathrm{sgn}(-\mathrm{j}s) = \mathrm{sgn}(\omega - \mathrm{j}\sigma) = \mathrm{sgn}(\omega)$, $H(s) = -\mathrm{j}\,\mathrm{sgn}(\Im[s])$.

Given $f(t)=\cos(\beta t)$, we have, $F(s) = \pi\{\xi(s-\mathrm{j}\beta) + \xi(s+\mathrm{j}\beta)\}$, $F_{\mathrm{Hi}}(s) = -\mathrm{j}\pi\,\mathrm{sgn}(\Im[s])\{\xi(s-\mathrm{j}\beta)+\xi(s+\mathrm{j}\beta)\} = -\mathrm{j}\pi\{\xi(s-\mathrm{j}\beta)-\xi(s+\mathrm{j}\beta)\}$, $f_{\mathrm{Hi}}(t) = \sin(\beta t)$.

Given $f(t)=\Pi_T(t)$, where $\Pi_T(t)$ is the centered rectangle of total width $2T$, we write $F(s)=2T\mathrm{Sa}(-\mathrm{j}Ts)=2T\mathrm{Sah}(Ts)$, where $\mathrm{Sah}(x)=\sinh(x)/x$ (Corinthios 1996, 2001). Hence,

$$F_{\mathrm{Hi}}(s) = -\mathrm{j}\,\mathrm{sgn}(\Im[s])2T\,\mathrm{Sah}(Ts) = \begin{cases} -\mathrm{j}2T\,\mathrm{Sah}(Ts), & \Im[s] > 0, \\ \mathrm{j}2T\,\mathrm{Sah}(Ts), & \Im[s] < 0. \end{cases}$$

9. Generalizing the discrete Hilbert transform

In the discrete-time domain, Hilbert transformer may be viewed as a filter of transfer function $H(\mathrm{e}^{\mathrm{j}\Omega}) = \begin{cases} -\mathrm{j}, & 0 < \Omega < \pi, \\ \mathrm{j}, & -\pi < \Omega < 0. \end{cases}$ To generalize the discrete Hilbert transform, we may write $\mathrm{sgn}[n] = 2u[n] - 1$,

$$\operatorname{sgn}[n] \overset{z}{\leftrightarrow} 2\left\{\frac{1}{1-z^{-1}} + \pi\psi(z)\right\} - 2\pi\psi(z) = \frac{2}{1-z^{-1}}.$$

The filter impulse response is

$$h[n] = \frac{1}{2\pi}\left\{\int_{-\pi}^{0} \mathrm{j}e^{\mathrm{j}\Omega n}\mathrm{d}\Omega + \int_{0}^{\pi} -\mathrm{j}e^{\mathrm{j}\Omega n}\mathrm{d}\Omega\right\} = \frac{1}{2\pi}\left\{\frac{1-e^{-\mathrm{j}\pi n}}{n} - \frac{e^{\mathrm{j}\pi n}-1}{n}\right\},$$

$$h[n] = \frac{1}{2n\pi}\{2-2\cos(\pi n)\} = \frac{1-\cos(\pi n)}{\pi n} = \begin{cases} 0, & n \text{ even}, \\ \frac{2}{\pi n}, & n \text{ odd}, \end{cases}$$

$H(z) = -\mathrm{j}\,\operatorname{sgn}(-\mathrm{j}\ln z)$. In other words, if $z=re^{\mathrm{j}b}$, $H(z)=-\mathrm{j}\operatorname{sgn}(b-\mathrm{j}\ln r)=-\mathrm{j}\operatorname{sgn}(b)$. Given a sequence $x[n]$, its Hilbert transform in the z domain is therefore $X_{\mathrm{Hi}}(z) = -\mathrm{j}\,\operatorname{sgn}(-\mathrm{j}\ln z)X(z)$, and in the time domain it is $x_{\mathrm{Hi}}[n]=x[n]^{*}h[n]$ where * means convolution.

With $x[n]=\cos(bn)$, $X(z)=\pi\{\psi(e^{-\mathrm{j}b}z)+\psi(e^{\mathrm{j}b}z)\}$,

$$X_{\mathrm{Hi}}(z) = -\mathrm{j}\operatorname{sgn}(-\mathrm{j}\ln z)X(z) = -\mathrm{j}\pi\operatorname{sgn}(-\mathrm{j}\ln z)\{\psi(e^{-\mathrm{j}b}z) + \psi(e^{\mathrm{j}b}z)\}.$$

Hence, $X_{\mathrm{Hi}}(z)=-\mathrm{j}\pi\{\psi(e^{-\mathrm{j}b}z)-\psi(e^{\mathrm{j}b}z)\}$ and $x_{\mathrm{Hi}}[n]=\sin(bn)$.

10. Generalized Hartley transform

Proposed by R. V. L. Hartley in 1942, the Hartley transform is closely related to the Fourier transform and is often favoured in applications involving imaging, X-ray diffraction, antennas and optical diffraction pattern analysis (Bracewell 1986, 2000, 2003). In fact, it transforms real images into real images and can thus be displayed wholly as one image, in contrast with the complex-valued Fourier transform. The Hartley transform of a function $f(t)$ which will be denoted $F_{\mathrm{Ha}}(\mathrm{j}\omega)$, being a special type of a Fourier transform, is given by

$$F_{\mathrm{Ha}}(\mathrm{j}\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\operatorname{cas}(\omega t)\mathrm{d}t,$$

where $\operatorname{cas}(\omega t) = \sin\omega t + \cos\omega t$. The inverse Hartley transform is given by

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_{\mathrm{Ha}}(\mathrm{j}\omega)\operatorname{cas}(\omega t)\mathrm{d}t.$$

The Hartley transform may be generalized by an extension in the Laplace plane. We may write

$$F_{\mathrm{Ha}}(s) = \frac{1}{2}\int_{-\infty}^{\infty} f(t)\{e^{st} + e^{-st} - \mathrm{j}(e^{st} - e^{-st})\}\mathrm{d}t$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(t)\{2\cosh st - \mathrm{j}2\sinh st\}\mathrm{d}t,$$

$$F_{\mathrm{Ha}}(s) = \int_{-\infty}^{\infty} f(t)\{\cosh st - \mathrm{j}\sinh st\}\mathrm{d}t.$$

The inverse transform is given by

$$f(t) = \frac{1}{4\pi\mathrm{j}}\int_{\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} F_{\mathrm{Ha}}(s)\{e^{st} + e^{-st} - \mathrm{j}(e^{st} - e^{-st})\}\mathrm{d}s$$

$$= \frac{1}{4\pi\mathrm{j}}\int_{\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} F_{\mathrm{Ha}}(s)\{2\cosh st - \mathrm{j}2\sinh st\}\mathrm{d}s,$$

$$f(t) = \frac{1}{2\pi\mathrm{j}}\int_{\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} F_{\mathrm{Ha}}(s)\{\cosh st - \mathrm{j}\sinh st\}\mathrm{d}s.$$

With $f(t)=e^{\alpha t}$,

$$F_{\mathrm{Ha}}(s) = \int_{-\infty}^{\infty} e^{\alpha t}\{\cosh st - \mathrm{j}\sinh st\}\mathrm{d}t = \int_{-\infty}^{\infty} e^{\alpha t}\left(\frac{e^{st}+e^{-st}}{2} - \mathrm{j}\frac{e^{st}-e^{-st}}{2}\right)\mathrm{d}t$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{(s+\alpha)t} + e^{-(s-\alpha)t} - \mathrm{j}e^{(s+\alpha)t} + \mathrm{j}e^{-(s-\alpha)t}\mathrm{d}t$$

$$= \frac{1}{2}\{2\pi\xi(-s-\alpha) + 2\pi\xi(s-\alpha) - \mathrm{j}2\pi\xi(-s-\alpha) + \mathrm{j}2\pi\xi(s-\alpha)\}$$

$$= \pi\{(1-\mathrm{j})\xi(s+\alpha) + (1+\mathrm{j})\xi(s-\alpha)\}.$$

11. Generalized discrete Hartley transform

The discrete Hartley transform (DHT) was introduced by Bracewell in 1983 (Bracewell 1983, 1986, 2003). It is related to the continuous-time domain Hartley transform in the same way the discrete Fourier transform (DFT) is related to the continuous-time domain Fourier transform. Given a sequence of N values, $x[0]$, $x[1],\ldots,x[N-1]$, the DHT denoted as $X_{\mathrm{Ha}}[k]$ is given by

$$X_{\mathrm{Ha}}[k]=\sum_{n=0}^{N-1}\operatorname{cas}(kn2\pi/N)x[n]=\sum_{n=0}^{N-1}\{\cos(kn2\pi/N)+\sin(kn2\pi/N)\}x[n].$$

The inverse DHT is given by

$$x[n]=\frac{1}{N}\sum_{n=0}^{N-1}\operatorname{cas}(kn2\pi/N)X_{\mathrm{Ha}}[k].$$

We note that the DFT of $x[n]$ is given by

$$X[k]=\sum_{n=0}^{N-1}e^{-j(2\pi/N)nk}x[n]=\sum_{n=0}^{N-1}\left\{\cos\left(\frac{2\pi}{N}nk\right)-j\sin\left(\frac{2\pi}{N}nk\right)\right\}x[n].$$

With $x[n]$ real, we may write $X_{\mathrm{Ha}}[k]=\Re\{X[k]\}-\Im\{X[k]\}$. If $x[n]$ has even symmetry, i.e. $x[N-n]=x[n],\quad n=1,2,\ldots,N-1$, then, $X[k]$ is real and $X_{\mathrm{Ha}}[k]=X[k]$. If $x[n]$ has odd symmetry, i.e. $x[N-n]=-x[n]$, $n=1$, $2,\ldots,N-1$ and $x[0]=0$, then $X[k]$ is pure imaginary and $X_{\mathrm{Ha}}[k]=jX[k]$. A generalized Hartley transform extending the transform over the complex z-plane may be written in the form

$$X_{\mathrm{Ha}}(z)=\frac{1}{2}\sum_{n=-\infty}^{\infty}x[n]\{(1+j)z^{-n}+(1-j)z^{n}\}=\frac{1}{2}\{(1+j)X(z)+(1-j)X(z^{-1})\}.$$

Let $x[n]=1$, $X(z)=2\pi\psi(z)$,

$$X_{\mathrm{Ha}}(z)=\frac{1}{2}\{(1+j)2\pi\psi(z)+(1-j)2\pi\psi(z^{-1})\}=2\pi\psi(z)=X(z).$$

Let $x[n]=a^{n}$, $X(z)=2\pi\psi(a^{-1}z)$,

$$X_{\mathrm{Ha}}(z)=\frac{1}{2}\{(1+j)2\pi\psi(a^{-1}z)+(1-j)2\pi\psi(a^{-1}z^{-1})\},$$

i.e.

$$X_{\mathrm{Ha}}(z)=\pi\{\psi(a^{-1}z)+\psi(az)+j[\psi(a^{-1}z)-\psi(az)]\}$$

.

12. Generalization of the Mellin transform

The Mellin transform of a causal function $f(x)$ is written in the form (Bracewell 2000; Gradshteyn & Ryzhik 2000; Poularikas 2000)

$$F_{\mathcal{M}}(s)=\int_0^{\infty}f(x)x^{s-1}\mathrm{d}x.$$

Let $x=e^{-t}$, $\mathrm{d}x=-e^{-t}\mathrm{d}t$, $x^{s-1}=e^{-t(s-1)}=e^{-ts}e^{t}$,

$$F_{\mathcal{M}}(s)=-\int_{\infty}^{-\infty}f(e^{-t})e^{-st}\mathrm{d}t=\int_{-\infty}^{\infty}f(e^{-t})e^{-st}\mathrm{d}t=\mathcal{L}[f(e^{-t})].$$

We note that the Millen transform of the function $f(x)$ is equivalent to the bilateral Laplace transform of the function $f(e^{-t})$. Since the bilateral Laplace transform of the most basic of functions does not exist, a great barrier has hitherto blocked the way to evaluating the Mellin transform of a large class of functions. Now, with the introduction of the generalized distributions and generalized Dirac-delta impulse, bilateral Laplace transforms of most functions can be readily obtained. The effect is opening the doors to the expansion of the domains of existence of the Mellin transform.

Given $f(x)=x^{j\beta}$, we have $f(e^{-t})=e^{-j\beta t}$,

$x_c(t)$	extended ξ transform $X_c(s)$
1	$2\pi\xi(s)$
e^{at}	$2\pi\xi(s-a)$
$\cosh(at)$	$\pi\{\xi[s-a]+\xi[s+a]\}$
$\cosh(j\beta t)$	$\pi\{\delta[\omega-\beta]+\xi[\omega+\beta]\}$
$u(t)$	$(1/s)+\pi\xi(s)$
$e^{at}u(t)$	$1/(s-a)+\pi\xi(s-a)$
$e^{at}\cos(\beta t)$	$\pi\{\xi[s-(\alpha+j\beta)]+\xi[s-(\alpha-j\beta)]\}$
$e^{at}\cos(\beta t)u(t)$	$(s-\alpha)/((s-\alpha)^2+\beta^2)+(\pi/2)\{\xi[s-(\alpha+j\beta)]+\xi[s-(\alpha-j\beta)]\}$
t	$-2\pi d\xi(s)/ds$
t^n	$(-1)^n 2\pi\xi^{(n)}(s)$
$t^n u(t)$	$n!/s^{n+1}+(-1)^n\pi\xi^{(n)}(s)$
$t^n e^{at}u(t)$	$(n!/(s-a)^{n+1})+(-1)^n\pi\xi^{(n)}(s-a)$
$1/(jt)+\pi\delta(t)$	$2\pi v(-s)$
$4\cos\beta t\cosh\alpha t$	$2\pi\{\xi(s-a)+\xi(s-a^*)+\xi(s+a)+\xi(s+a^*)\}$
$(-t)^n$	$2\pi\xi^{(n)}(s)$
$-tu(-t)$	$(1/s^2)+\pi\xi'(s)$
$t^n e^{at}u(t)$	$(n!/(s-\alpha)^{n+1})+(-1)^n\pi\xi^{(n)}(s-\alpha)$
$(-t)^n e^{-at}u(-t)$	$n!/[(-1)^{n+1}(s+\alpha)^{n+1}]+\pi\xi^{(n)}(s+\alpha)$

With $f(x)=x^{-j\beta}$ $F_{\mathcal{M}}(s)=\mathcal{L}[e^{i\beta t}]=2\pi\xi(s-j\beta)$. With $f(x)=x^{j\beta}+x^{-j\beta}$,

$F_{\mathcal{M}}(s) = 2\pi\{\xi(s+j\beta)+\xi(s-j\beta)\}$.
With $f(x)=x^{j\beta}-x^{-j\beta}$,
$F_{\mathcal{M}}(s) = 2\pi\{\xi(s+j\beta)-\xi(s-j\beta)\}$.

13. Application to multidimensional signals and the solution of differential equations

The extended generalized transforms are applicable to the transformation of multidimensional signals. As an example of such applications, new extended two-dimensional bilateral z transforms of some basic sequences. As an example of the application of extended transforms to partial differential equations, we consider the solution of the heat equation $\frac{\partial u(x,t)}{\partial t}-\frac{\partial^2 u(x,t)}{\partial x^2}=te^{\alpha t}$, with the boundary conditions $u(0,t)=u(1,t)=0$ and the initial condition $u(x,0)=0$. Laplace transforming both sides of the partial differential equation, we have $\frac{d^2U(x,s)}{dx^2}-sU(x,s)=-2\pi\xi'(s-\alpha)$.
The particular solution has the form $U_p(x,s)=A_0$ which upon substitution in the equation implies that $-sA_0=-2\pi\xi'(s-\alpha)$, i.e. $A_0=2\pi\xi'(s-\alpha)/s=(2\pi/\alpha)\xi'(s-\alpha)+(2\pi/\alpha^2)\xi(s-\alpha)$, and the general solution has the form $U(x,s)=k_1\cosh\sqrt{s}x+k_2\sinh\sqrt{s}x+\frac{2\pi}{\alpha}\xi'(s-\alpha)+\frac{2\pi}{\alpha^2}\xi(s-\alpha)$.
Using the initial condition $U(0,s)=U(1,s)=0$ and $U(1,s)=0$, we have

$$k_1=-\frac{2\pi}{\alpha}\xi'(s-\alpha)-\frac{2\pi}{\alpha^2}\xi(s-\alpha),$$

$$k_1\cosh\sqrt{s}+k_2\sinh\sqrt{s}+\frac{2\pi}{\alpha}\xi'(s-\alpha)+\frac{2\pi}{\alpha^2}\xi(s-\alpha)=0,$$

$$k_2=\frac{\left[-k_1\cosh\sqrt{s}-\frac{2\pi}{\alpha}\xi'(s-\alpha)-\frac{2\pi}{\alpha^2}\xi(s-\alpha)\right]}{\sinh\sqrt{s}}.$$

We obtain

$$U(x,s)=(2\pi/\alpha)F(x,s)\xi'(s-\alpha)+(2\pi/\alpha^2)F(x,\alpha)\xi(s-\alpha),$$

where

$$F(x,s)=1+\coth\sqrt{s}\sinh\sqrt{s}x-\sinh\sqrt{s}x/\sinh\sqrt{s}-\cosh\sqrt{s}x.$$

Since, in general, $F(s)\xi'(s-\alpha)=F(\alpha)\xi'(s-\alpha)-F'(\alpha)\xi(s-\alpha)$, we may write $U(x,s)=(2\pi/\alpha)F(x,\alpha)\xi'(s-\alpha)-(2\pi/\alpha)\{F'(x,\alpha)-F(x,\alpha)/\alpha\}\xi(s-\alpha)$, which can be written in the form $U(x,s)=(2\pi/\alpha)F(x,\alpha)\xi'(s-\alpha)-(2\pi/\alpha)G(x,\alpha)\xi(s-\alpha)$, with $G(x,\alpha)=F'(x,\alpha)-F(x,\alpha)/\alpha$.
After some algebraic manipulation, we obtain

$$F(x,\alpha)=-2\sinh\left[\frac{(\sqrt{\alpha}x-\sqrt{\alpha})}{2}\right]\sinh\left[\sqrt{\alpha}x/2\right]/\cosh\left[\sqrt{\alpha}/2\right],$$

and

$$G(x,\alpha)=\frac{-1}{4\alpha}\{\operatorname{sech}^2(\sqrt{\alpha}/2)[2+2\cosh\sqrt{\alpha}-2\cosh(\sqrt{\alpha}x-\sqrt{\alpha})-2\cosh\sqrt{\alpha}x+\sqrt{\alpha}x\sinh(\sqrt{\alpha}x-\sqrt{\alpha})-\sqrt{\alpha}\sinh\sqrt{\alpha}x+\sqrt{\alpha}x\sinh\sqrt{\alpha}x]\},$$

whence the solution

New two-dimensional bilateral z transforms. $S(n,m)\equiv S_n^{(m)}=(1/m!)\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^n$ are the Stirling numbers of the second kind.

14. Conclusion

New extended Laplace, z, Hilbert, Hartley and Mellin transforms for one- and multidimensional signals are obtained using the recently introduced generalized distributions. The extended transforms are now well defined for a large class of functions from the most basic to highly complex fast-rising exponential ones that have so far had no transform. The paper illustrates clearly the significant impact of extending the domains of Laplace and z transforms on other transforms and mathematical tools, after nearly a century during which bilateral Laplace and z transforms of even the most basic of functions were undefined.

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