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A Study on Applications of Partial Differential Equation and Its Pelation with Fourier

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Abstract

The aim of this paper is to study the partial differential equation and its applications. The research also aims at studying the relation between partial differential equation and Fourier. The partial differential equation forms are derived below,

(1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional wave equation*

(2) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional heat equation*

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *Two-dimensional Laplace equation*

(4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ *Two-dimensional Poisson equation*

The study varies solution and examples of solving partial differential equations are also derived. And the relation between Fourier series and transformation for partial differential equations is presented.

Ordinary Differential Equations

A differential equation, simply put, is an equation involving one or more derivatives of a function y = f(x). These equations can be as straightforward as

(1) $y' = 3$,

or more complicated, such as

(2) $y'' + 12y = 0$

or

(3) $(x^2y''') + e^x y' - 3xy = (x^3 + x)$.

There are a number of ways of classifying such differential equations. At the least, you should know that the **order** of a differential equation refers to the highest order of derivative that appears in the equation. Thus these first three differential equations are of order 1, 2 and 3 respectively (Webster, A. G. (1955)).

Differential equations show up surprisingly often in a number of fields, including physics, biology, chemistry and economics. Anytime something is known about the rate of change of a function, or about how several variables impact the rate of change of a function, then it is likely that there is a differential equation hidden behind the scenes.

Many laws of physics take the form of differential equations, such as the classic force equals mass times acceleration (since acceleration is the second derivative of position with respect to time).

Modeling means studying a specific situation to understand the nature of the forces or relationships involved, with the goal of translating the situation into a mathematical relationship.

It is quite often the case that such modeling ends up with a differential equation. One of the main goals of such modeling is to find solutions to such equations, and then to study these solutions to provide an understanding of the situation along with giving predictions of behavior.

In biology, for instance, if one studies populations (such as of small one-celled organisms), and their rates of growth, then it is easy to run across one of the most basic differential equation models, that of **exponential growth**. To model the population growth of a group of e-coli cells in a Petri dish, for example, if we make the assumption that the cells have unlimited resources, space and food, then the cells will reproduce at a fairly specific measurable rate. The trick to figuring out how the cell population is growing is to understand that the number of new cells created over any small time interval is proportional to the number of cells present at that time (Strauss, W. A. (1992)).

This means that if we look in the Petri dish and count 500 cells at a particular moment, then the number of new cells being created at that time should be exactly five times the number of new cells being created if we had looked in the dish and only seen 100 cells (i.e. five times the population, five times the number of new cells being created). Curiously, this simple observation led to population studies about humans (by Malthus and others in the 19th century), based on exactly the same idea of proportional growth. Thus, we have a simple observation that the rate of change of the population at any particular time is in direct proportion to the number of cells present at that time. If you now translate this observation into a mathematical statement involving the population function y = P(t), where tstands for time, and P(t) is the function giving the population of cells at time t then you have become a **mathematical modeler**.

Answer (yielding another example of a differential equation):

(3) $P'(t) = k\,P(t)$, or, equivalently, $y' = k\,y$

To **solve** a differential equation means finding a solution function $y = f(t)$, such that when the corresponding derivatives, y', y'', etc. are computed and substituted into the equation, then the equation becomes an identity (such as "$3x = 3x$").

For instance, in the first example, equation (1) from above, the differential equation $y' = 3$ is equivalent to the condition that the derivative of the function $y = f(x)$ is a constant, equal to 3. To solve this means to find a function whose derivative with respect to x is constant, and equal to 3. Many such functions come to mind quickly, such as $y = 3x$, or $y = 3x + 5$, or $y = 3x – 12$. Each of these functions is said to **satisfy** the original differential equation, in that each one is a specific solution to the equation. In fact, clearly anything of the form $y = 3x + c$, where c is any constant, will be a solution to the equation. And on the other hand, any function that actually satisfies equation (1) will have to be of the form $y = 3x + c$.

To separate the idea of a specific solution, such as $y = 3x + 5$, from a more general set or **family** of solutions, $y = 3x + c$, where c is an arbitrary constant, we call an individual solution function, such as asy $= 3x + 5$, a **specific** or **particularsolution** (no surprise there). Next we call the set of functions, $y = 3x + c$, which contain an arbitrary constant (or constants), a **general solution**.

Note that if someone asks you to come up with a solution to the differential equation $y' = 3$, then any function of the form $y = 3x + c$, would do as an answer. However, if the same person asked you to solve $y' = 3$, and at the same time make sure that the solution also satisfies another condition, such as $y(0) = 20$, then this extra condition forces the constant c to equal 20.

To see this, note that the general solution is $y = 3x + c$. To find $y(0)$, this just means substituting in $x = 0$, so that you find that $y(0) = c$. Then to make the result equal to 20, according to the extra condition, it must be the case that the constant c equals 20, so that the particular solution in this situation is $y = 3x + 20$, now with no arbitrary constants remaining (Süli, E., &Mayers, D. F. (2003)).

An extra condition in addition to a differential equation, is often called an **initial condition**, if the condition involves the value of the function when $x = 0$ (or $t = 0$, or whatever the independent variable is labeled for the function in the given situation). Sometimes to identify a specific solution to an ODE, several initial conditions need to be given, not just about the value of the function, y = f(x), when x = 0, but also giving the value of the first derivative of y when x = 0, or even the value of higher order derivatives as well.

This will often happen, for instance, if the differential equation itself involves derivatives of higher order. In fact, typically the order of the highest derivative that shows up in the equation will equal the number of constants that show up in the general solution to a differential equation. You can picture why this happens by thinking of the process of solving the differential equation as involving as many integrations as the highest order of derivative, to "undo" each of the derivatives, then each such indefinite integration will bring in a new constant.

Solutions To Partial Differential Equations:

To solve an ordinary differential equation, one seeks specific functions, such as $\sin(\alpha t)$ or r^n that satisfy the equation. In contrast, solutions to partial differential equations are determined by the argument of the functional form (Webster, A. G. (1955)). For example, in Equation 3, any function whose argument is $\eta \equiv x^2/(2Dt)$ is a solution to the equation since, regardless of the form for $f(\eta)$,

$$\frac{\partial \eta}{\partial t} = -\frac{1}{D}\frac{x^2}{t^2};\ \frac{\partial \eta}{\partial x} = \frac{2x}{Dt}$$

$$\frac{\partial f(\eta)}{\partial t} = \frac{\partial f(\eta)}{\partial \eta}\frac{\partial \eta}{\partial t} = f'(\eta)\frac{-x^2}{Dt^2};\ \frac{\partial f(\eta)}{\partial x} = \frac{\partial f(\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} = f'(\eta)\frac{2x}{Dt};\ \frac{\partial^2 f(\eta)}{\partial x^2} = f''(\eta)\frac{4x^2}{D^2t^2}$$

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2} \qquad \text{Eq.4}$$

The first objective in solving a partial differential equation is to find a way to convert the partial derivatives to total derivatives, which means that the derivatives must operate on a function of a single variable only (Strauss, W. A. (1992)). In this course, three methods will be used for solving partial differential equations.

Separation of Variables:

For separation of variables, the function to be found is written as a product of two or more functions, each of which depends on only one variable. Thus, if one seeks the solution for a velocity $v(r,\theta,\phi)$, where v is a velocity component, one writes that $v(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$. This assumption is restrictive in that many functions cannot be written in this form. For example, while the function $v(r,\theta,\phi) = r^2\cos\theta\sin\phi$ is in such a form (with $R(r) = r^2$, $\Theta(\theta) = \cos\theta$, and $\Phi(\phi) = \sin\phi$), a function such as $v(r,\theta,\phi) = r\sqrt{\sin\theta + \cos\phi}$ cannot be written in such a manner (Strang, G., &Aarikka, K. (1986)).

Exercise 1: Which of the following forms is not separable?

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a. $f(x,y)=xy$

b. $f(x,y)=\sqrt{xy}$

c. $f(x,y)=\sqrt{x+y}$

d. $f(r,\theta)=e^{r\theta}$

e. $f(r,\theta)=e^{r+\theta}$

f. $f(r,\theta l,\phi)=r^2\sin(\theta+\phi)$

g. $f(r,z,\theta)=r^2(1-z^2)\sin(\alpha\theta^3)$

h. $f(r,z,\theta)=r^2+zr\cos(\theta)$

Exercise 2: For each of the separable forms in Exercise 1, what are the separated functions? (Example: for $f(x,y)=x/y$, the separated functions are $X(x)=x$ and $Y(y)=y^{-1}$).

If a solution to a partial differential equation is separable, then the boundary conditions must be separable. To illustrate the method of separation of variables, the following equation will be considered:

$$\nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0 \qquad \text{Eq. 3}$$

with the boundary conditions:

$$\begin{aligned} &f(0,y)=0\\ &f(1,y)=g(y)\\ &f(x,0)=0\\ &f(x,1)=0 \end{aligned} \qquad \text{Eq. 4}$$

The function $g(y)$ will not yet be specified, but in general it is given an explicit form in the statement of the problem. The boundary conditions are separable. The substitution $f(x,y)=X(x)Y(y)$ is made in the differential equation so that:

$$\nabla^2 f \equiv \frac{\partial^2 X(x)Y(y)}{\partial x^2}+\frac{\partial^2 X(x)Y(y)}{\partial y^2}=0$$

Since $Y(y)$ is not a function of x, it can be treated as a constant with respect to the derivative in x. Similarly, since $X(x)$ is not a function of y, it can be treated as a constant with respect to the derivative in y. Therefore:

$$\nabla^2 f \equiv Y(y)\frac{\partial^2 X(x)}{\partial x^2}+X(x)\frac{\partial^2 Y(y)}{\partial y^2}=0 \Rightarrow Y(y)\frac{\partial^2 X(x)}{\partial x^2}=-X(x)\frac{\partial^2 Y(y)}{\partial y^2}.$$

Next, the equation is divided by the product $X(x)Y(y)$ to yield.

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{dx^2}=\frac{-1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}.$$

The left hand side of this equation is a function of x only, and the right hand side of the equation is a function of y only (Hawken, D. F., Gottlieb, J. J., & Hansen, J. S. (1991)). But if the left hand side changes as x changes, it is not possible for the right hand side to change because it does not vary with x. Similarly, it is not possible for the left hand side to change as the right hand side changes with y. Thus, the only way the two sides can be equal is if they are both independent of x and y. They must therefore be equal to some constant k. It follows that:

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}=k$$

$$\frac{-1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}=k, \text{ or } \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}=-k$$

In addition, since the functions $X(x)$ and $Y(y)$ are functions of x only and y only, respectively, it is now possible to replace the partial derivative with an ordinary derivative so that

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$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=k$$
$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=-k$$

Each equation can now be multiplied by $X(x)$ or $Y(y)$, as appropriate and the two *ordinary* differential equations to be solved are:

$$\frac{d^2X(x)}{dx^2}-kX(x)=0$$
$$\frac{d^2Y(y)}{dy^2}+kY(y)=0$$

The solution to the equation in x is

$$X(x)=A_x\cosh\left(\sqrt{k}x\right)+B_x\sinh\left(\sqrt{k}x\right)$$

and the solution to the equation in y is

$$Y(y)=A_y\cos\left(\sqrt{k}y\right)+B_y\sin\left(\sqrt{k}y\right)$$

The boundary conditions at $x=0$ and $y=0$ require that $A_x=A_y=0$. Furthermore, the boundary condition for $y=1$ requires that $B_y\sin\left(\sqrt{k}\right)=0$, so that $\sqrt{k}=n\pi$, where n is any integer value from $-\infty$ to $+\infty$. These are the eigenvalues of the equation. With these eigenvalues,

$$f_n(x,y)=X(x)Y(y)=B_x\sinh(n\pi x)B_y\sin(n\pi y)=C_n\sinh(n\pi x)\sin(n\pi y),$$

where the two constants B_x and B_y have been combined into the single constant C_n. The subscript n in $f_n(x,y)$ designates that this is the solution corresponding to a specific eigenvalue n.

Now the boundary condition for $x=1$ must be satisfied. In the simplest case, where $g(y)$ has the form $A\sin(m\pi y)$ (with some specified constants A and m), the solution is readily obtained with $C=A$ and $n=m$. It is not likely, however, that we will be lucky enough to enjoy this result. Therefore, we must consider the more general case of an arbitrary $g(y)$. Note that $g(y)$ must satisfy the boundary conditions at $x=0$ and $x=1$ or else it will not be a valid boundary condition for this problem (Thompson, et al., (1982)).

Through the process of separation of variables, we have found a family of solutions to the differential equations. Because the differential equation is linear, any linear combination of these solutions is also a solution to the equation. Specifically, we can write:

$$f(x,y)=\sum_{n=-\infty}^{\infty}f_n(x,y)=\sum_{n=-\infty}^{\infty}C_n\sinh(n\pi x)\sin(n\pi y).$$

This is then evaluated at $x=1$, using the last boundary condition to evaluate the coefficients C_n. The following equation is obtained:

$$g(y)=\sum_{n=-\infty}^{\infty}C_n\sinh(n\pi)\sin(n\pi y)$$

$$\int_{-\pi}^{\pi}\sin(n\pi x)\sin(m\pi x)dx=0 \quad if \quad m\neq n$$
$$\pi \quad if \quad m=n$$

Students who are familiar with the techniques used to derive Fourier series will immediately recognize the method required to determine the value of C_n for a given value of n. First, multiply the equation by $\sin(m\pi y)$. Then integrate both sides from $-\pi$ to π. One obtains:

$$\int_{-\pi}^{\pi}g(y)\sin(m\pi y)dy=\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}C_n\sinh(n\pi)\sin(n\pi y)\sin(m\pi y)dy.$$

The integral and summation on the right hand side can be interchanged, and since $C_n\sinh(n\pi)$ is independent of y,

$$\int_{-\pi}^{\pi}g(y)\sin(m\pi y)dy=\sum_{n=-\infty}^{\infty}C_n\sinh(n\pi)\int_{-\pi}^{\pi}\sin(n\pi y)\sin(m\pi y)dy$$

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But the sine wave is an orthogonal function so that the integral is zero unless $m = n$, in which case it has the value of π. Therefore:

$$\frac{1}{\pi \sinh(n\pi)} \int_{-\pi}^{\pi} g(y) \sin(n\pi y)\, dy = C_n .$$

If we substitute this result back into the functional form for $f(x, y)$, we find that

$$f(x,y) = \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{\pi \sinh(n\pi)} \int_{-\pi}^{\pi} g(y) \sin(n\pi y)\, dy \right\} \sinh(n\pi x) \sin(n\pi y)$$

ELEMENTS OF TRIGONOMETRIC FOURIER SERIES FOR SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

In this section we discuss Fourier series expassion of arbitrary, even and odd functions.

1. Example1

Let f (x) =x for $-\pi \le x \le \pi$. We will write the Fourier series of f on $[-\pi,\pi]$. The Fourier coefficients are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x\, dx = 0.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \operatorname{cox}(nx)\, dx.$$

$$\left[\frac{1}{n^2 \pi} \cos(nx) + \frac{x}{n\pi} \sin(nx) \right]_{-\pi}^{\pi} = 0,$$

and $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx$$

$$\left[\frac{1}{n^2 \pi} \sin(nx) - \frac{x}{n\pi} \cos(nx) \right]_{-\pi}^{\pi}$$

$$= -\frac{2}{n} \cos(n\pi) = \frac{2}{n} (-1)^{n+1},$$

sincecos $(n\pi) = (-1)^n$ if n is an integer. The Fourier series of x on $[-\pi,\pi]$ is

$$\sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(nx) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x) - ..$$

In this example, the constant term and cosine coefficients are all zero, and the Fourier series contains only sine terms.

1.1. Example 2

Let

$$f(x) = \begin{cases} 0 & \text{for } -3 \le x \le 0 \\ x & \text{for } 0 \le x \le 3 \end{cases}$$

Here l = 3 and the Fourier coefficients are

$$a_0 = \frac{1}{3} \int_{-3}^{3} f(x)\, dx = \frac{1}{3} \int_{0}^{3} x\, dx = \frac{3}{2}$$

$$a_n = \frac{1}{3} \int_{-3}^{3} f(x) \cos\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{1}{3} \int_{0}^{3} x \cos\left(\frac{n\pi x}{3}\right) dx$$

$$= \left. \frac{3}{n^2\pi^2} \cos\left(\frac{n\pi x}{3}\right) + \frac{x}{n\pi} \sin\left(\frac{n\pi x}{3}\right) \right]_0^3$$

$$= \frac{3}{n^2\pi^2} [(-1)^n - 1]$$

and

$$bn = \frac{1}{3} \int_{-3}^{3} f(x) \sin\left(\frac{n\pi x}{3}\right) dx = \frac{1}{3} \int_{0}^{3} x \sin\left(\frac{n\pi x}{3}\right) dx$$

$$\left. \frac{3}{n^2\pi^2} \sin\left(\frac{n\pi x}{3}\right) - \frac{x}{n\pi} \cos\left(\frac{n\pi x}{3}\right) \right]_0^3$$

$$= \frac{3}{n\pi} (-1)^{n+1}$$

The Fourier series of f on [-3,3] is

$$\frac{3}{4} + \sum_{n=1}^{\infty} \left[\left(\frac{3}{n^2\pi^2} [(-1)^n - 1] \cos\left(\frac{n\pi x}{3}\right) + \frac{3}{n\pi} (-1)^{n+1} \sin\left(\frac{n\pi x}{3}\right) \right] \right.$$

CONCLUSION

In this work, when modeling problems over regions that extended very far in at least one direction, we often idealized the situation to that of a problem having infinite extent in one or more directions, where any boundary conditions that would have applied on the far-away boundaries are discarded in favor of simple bounded-ness conditions on the solution as the appropriate variable is sent to infinity. Such problems were mathematically modeled by differential equations defined on infinite regions. For one-dimensional problems we distinguish two types of infinite regions: infinite intervals extending from -∞ to ∞ and semi-infinite intervals extending from one point (usually the origin) to infinite (usually +∞) are infinite, but by introducing a mathematical model with infinite extent, we are able to determine behavior of problems in the situations in which the influence of actual boundaries are expected to be negligible. Thus the seminar paper developed the Fourier transform method and applied it to solve: heat flow problem of an infinitely long thin bar insulated on its lateral surface, heat flow in a semi–infinite region, wave equation, Laplace equation in a half-plane and in a semi-infinite strip, and some partial differential equation on the entire real line.

Even though a survey of this seminar paper shows that what is actually studied Fourier transform method to PDE is that, we taken the Fourier transform of PDE and its initial and boundary conditions to reduce it into an ODE. We then solved this ODE for the transformed function. We inverted this function to determine the solution to our PDE. This is not just a method that is specific to the Fourier transform because this method also works for the Laplace transform and in general for many integral transforms. The integrals defining the Fourier transform and its inverse are remarkably alike, and this symmetry was often exploited, for example when assembling appendix given for Fourier transforms. One condition on this is that the variable you taken to the integral transform its domain must match the range of integration of the integral transform. The type of boundary and initial conditions that are given should also played a role in which transform should be used. In case, the Fourier transform is used to analyze boundary value problems on the entire line. The extension of Fourier methods to the entire real line leads naturally to the Fourier transform, an extremely powerful mathematical tool for the analysis of non-periodic functions.

$$\frac{\partial g(x,t)}{\partial t} = c\frac{\partial^2 g(x,t)}{\partial x^2} \Rightarrow g_t = cg_{xx} \qquad \textbf{[Equation 9]}$$

$$h(x) = g(x,0) \qquad \textbf{[Equation 10]}$$

$$g(x,t) = F^{-1}\left\{H(f)e^{-4\pi^2 f^2 ct}\right\} = h(x) * \frac{e^{-x^2/(4ct)}}{\sqrt{4\pi ct}} = \int_{-\infty}^{\infty} h(z)\frac{e^{-(z-x)^2/(4ct)}}{\sqrt{4\pi ct}}dz \qquad \text{[Equation 1]}$$

The result in Equation [1] represents the general solution of Equation [9], subject to the condition of Equation [10]. The result, in essence, represents how the initial heat distribution smooths itself out over time. That is, as t approaches 0, g(x,t) approaches h(x). This is because the Gaussian function becomes very sharp and approximates an impulse function. As time increases, the integration of Equation [17] represents a "smoothing out" or averaging of the initial temperature distribution h(x). This equation dictates precisely how heat spreads out on a rod.The solution of a IBVP consisting of a partial differential equation together with boundary and initial conditions can be solved by the Fourier transform method. In one dimensional boundary value problems, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

Suppose that u(x,t) is a function of two variables x and t, where -∞<x<∞ and t>0 Because of the presence of two variables, care is needed in identifying the variable with respect to which the Fourier transform is computed. For example, for fixed t, the function u(x,t) becomes a function of the spatial variable x, and as such, we can take its Fourier transform with respect to the x variable. We denote this transform by

$$F(u(x,t))(\omega)=\hat{u}(\omega,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx$$

$\hat{u}(\omega,t)$ Thus,

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