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"Some Results on the Dual Space of Normed Almost Linear Space"

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Abstract: *G. Godini introduced the concept normed almost linear space which generalizes normed linear space. To support the idea that the normed almost linear space is a good concept the notion of a dual space of normed almost linear space X, has been introduced in this paper. In this paper we prove some results like if X is normed almost linear space then* i)V_{X^*} *is a Banach space,* ii) *If B is a basis of X then for each* $b_0 \in B\backslash V_X$ *there exists* $f \in X^{\#}$ *such that* $f(b_0) = 1$ *and* $f(b_0) = 0$ *for each* $b \in B\backslash\{b_0\}$. *If* $b_0 \in W_X$ *then* $f \in X^*$iii) *If* W_X *has a basis, then* $X \neq \{0\}$ iv)*If B is a basis such that* $card(B\backslash V_X) < \infty$, *then* $X^* = \{ f \in X^{\#} : f\backslash V_X \in (V_X)^* \}$ *and is total over X and*v) *If* $f \in (W_X)^*\}$,*then there exists* $f_1 \in X^*$ *such that* $f_1\backslash W_X = f$, $|||f_1||| = |||f|||$ *and* $f_1\backslash V_X = 0$. *Using these results we prove that if X is strong normed almost linear space such that ρ is a metric and if* $x \in X\backslash (W_X + V_X)$, $X = \{\alpha x_0 + \mu(-x_0) + w + v : w \in W_X,\ v \in V_X,\ \alpha, \mu \geq 0\}$ *then i) for each* $f \in (V_X)^*$ *there exists* $f_1 \in V^*$ *such that* $f_1 \backslash V_X = f$ *ii)* $V_X{}^* \neq \{0\}$ *and ii) for each* $f \in (W_X + V_X)^*$ *there exists* $f_1 \in X^*$ *such that* $f_1 \backslash (W_X + V_X) = f$.

Keywords: Almost linear space (als), basis of an almost linear space, almost linear functional,norm on an almost linear space,normed almost linear space (nals),strong normed almost linear spaces (snals),the dual space ofnormed almost linear space.

1. **Introduction:**G. Godini[2] introduced the concept normed almost linear space which generalizes normed linear space. All spaces involved in this chapter are over the real field ℝ. A normed almost linear space is an almost linear space X together with a functional $|||\ .\ ||| : X \to \mathbb{R}$ called a norm which satisfies all the axioms of an usual norm on a linear space as well as some addition ones i.e. $|||x - z||| \leq |||x - y||| + |||y - z|||$. Due to the fact that they have weakened the axioms of a linear space but they have strengthen the axioms of the norm. Since the norm of a normed almost linear space X does not generate a metric on$X$, they considered the strong normed almost linear space which also generalizes the normed linear space. To support the idea that the normed almost linear space is a good concept they introduced the concept of a dual space of normed almost linear space X where the functional on

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X are no longer linear but almost linear which is also a normed almost linear space. This chapter consists three sections.

2. Preliminaries:

2.1: Almost linear space (als): An almost linear space(als) is a non empty setXtogether with two mappings **s:** Xx$X \to$ X and m: ℝ x $X \to X$ satisfying (i) – (viii)below.

For x , y ∈X and α∈ℝ we denote s(x ,y) by $x + y$ and m(α , x) by αx.Let x, y, z ∈X and α ,β∈ℝ i) (x +y) + z = x + (y +z)ii) $x + y = y + x$ iii) There exists an element 0 ∈X such that x+ 0 = x for each x∈X. iv)α(x + y) = αx + α y v) (α +β) x = αx + βx for α≥0, β≥0 vi) α(βx) = αβ (x), vii) 1x = x and viii) 0x = 0.

For an almost linear space X we introduce the following two sets

$V_X = \{x \in X: x - x = 0\}andW_X = \{x \in X: x = -x\}$

2.2: Basis of an almost linear space: A subset B of an almost linear space X is called a basis of X if for each x∈X \{0} there existunique sets $\{b_1,\ldots,b_n\} \subset$ B and $\{\alpha_1,\ldots,\alpha_n\} \subset$ℝ \{0}(n depending on x) such that $x = \sum \alpha_i b_i$ (i=1,…,n) where $\alpha_i > 0$ for $b_i \notin V_X$.

2.2.3: Almost linear functional: Let X be an almost linear space. A function f:$X \to$ ℝ is called an almost linear functional if f satisfies the following conditions. For x, y∈Xandα(≥0) ∈ℝ

i) $f(x+y) = f(x) + f(y)$ ii) $f(\alpha x) = \alpha f(x)$ iii) $-f(-x) \leq f(x)$ (or) $f(w) \geq 0$ for every $w \in W_X$.

The set of all almost linear functional defined on an almost linear space X is denoted by $X^{\#}$.

2.4: Norm on an almost linear space: A norm ||| · ||| on an almost linear space X is a function satisfying the following conditions $N_1 - N_3$.

Let x, y, z ∈X and α∈ℝ**.** N_1.|||x ||| = 0 if and only if x=0.

N2.|||αx ||| = |α| |||x|||, and

N_3. |||x - z |||≤ ||| $x – y$ ||| +|||y - z |||

2.5: Normed almost linear space (nals): An almost linear space X together with |||**.** ||| **:**$X \to$ ℝ satisfying $N_1 – N_3$ is called a normed almost linear space .

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2.6: Metric: A metric on normed almost linear space d: Xx$X \to \mathbb{R}$ is defined as d(x, y) = $||| v_x - v_y + \sum (\alpha_i - \beta_i) b_i |||$, i=1 to n , $x, y \in X$, v_x , $v_y \in V_X$, $b_i \in B \backslash V_X$, α_i , $\beta_i \geq 0$.

2.7: Strong normed almost linear spaces (snals): A strong normed almost linear space is a normed almost linear space$X$ together with a semi-metric ρon X which satisfies the following conditions.

For $x, y, z \in$X and $\alpha \in \mathbb{R}$

i) $||| x ||| - ||| y ||| \leq \rho(x, y) \leq ||| x - y |||$

ii) $\rho(x+z, y+z) \leq \rho(x, y)$

iii) The function $\alpha \to \rho(\alpha x, x)$ is continuous at $\alpha = 1$

2.8: The dual space: Let $X^* = \{ f \in X^{\#} : |||f||| < \infty \}$, then the space X^* together with $||| . |||$ defined by $|||f||| = \sup\{ |f(x)| : |||x||| \leq 1 \}$is called the dual space of the normed almost linear space X.

Lemma 2.9: Let X be an almost linear space and let $f \in X^{\#}$. We have $f \in V_{X^{\#}}$ iff f is linear on X,if and only if $-1of = -f$, if and only if $f/W_X = 0$.

Lemma 2.10: Let X be an almost linear space with a basis B, then the sets $\{-b : b \in B\}$ and$\{\alpha_b b : b \in B, \alpha_b \neq 0, \alpha_b > 0 \text{ for } b \notin V_X\}$ are also bases of X.

Corollary 2.11: Let X be an almost linear space with a basis B then W_X has a basis.

Proof: Let B be a basis of X.

Theorem 2.12: Let B be a basis of the almost linear space X. Then there exist a basis $B'$of X with the property that for each $b' \in B' \backslash V_X$ we have $-b' \in B' \backslash V_X$. Moreover card $(B \backslash V_X) = card(B' \backslash V_X)$.

Lemma2.13: Let X be a normed almost linear space and let$x \in X$,$w \in W_X$, then$max\{ |||x|||, |||w||| \} \leq |||x + w|||$.

Lemma2.14: Let X be a normed almost linear space and let$x, x_n \in X$, $\alpha_n \in R$,$n \in N$,$\lim \alpha_n = \infty$. If the sequence$\{||| \alpha_n x + x_n |||\}_{n=1}^{\infty}$is bounded, then$x \in V_X$.

3. **Some results:**

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Theorem3.1: Let X be a nals and for $f \in X^{\#}$define $||| f|||= sup\ \{|f(x)|: x \in X||| x|||\leq 1\}$. Let $X^* = \{f \in X^{\#}: ||| f||| < \infty\}$, then X^*together with $|||\ .\ |||$, defined as above is a normed almost linear space.**Proof:** It is easy to show that $X^*$is an almost linear space. We now show that the $|||\ .|||$ defined in hypothesis satisfies the conditions$N_1 - N_3$.

We now show N_1. To show N_1we have to show that for $f_i \in X^*, i = 1,2,3$.

$||| f_1 + (-1of_3)||| \leq |||f_1 + (-1of_2)||| + |||f_2 + (-1of_3)|||$.

Let $x \in B_X$, then $|f_1 + (-1of_3)(x)| = |f_1(x) + f_3(-x)|$.

If$|f_1 + (-1of_3)(x)| = -f_1(x) - f_3(-x)$,

Then by definition of almost linear functional we get

$$|f_1 + (-1of_3)(x)| = -f_1(x) - f_3(-x) \leq f_1(-x) + f_2(x) + f_2(-x) + f_3(x)$$

$$\leq |f_1 + (-1of_2))(-x)| + |(f_2 + (1of_3))(-x)|$$

$$\leq |||f_1 + (-1of_2)||| + |||f_2 + (-1of_3)|||$$

If$|f_1 + (-1of_3))(x)| = |f_1(x) + f_3(-x)|$.

In this case also we get

$|f_1 + (-1of_3))(x)| \leq ||| f_1 + (-1of_2) ||| + ||| f_2 + (-1of_3)|||$

To showN_2 we have to show that $||| \alpha f ||| = |\alpha | |||f|||$.

Now for some $\alpha \in \mathbb{R}$we have

$||| \alpha f ||| = sup\ \{| \alpha f(x)| : ||| x ||| \leq 1\} = |\alpha| sup\{|f(x)|: ||| x ||| \leq 1\} = |\alpha||||f|||$.

N_3follows trivially.■

Proposition3.2: For any normed almost linear space X, the dual space X^* is a strong normed almost linear space for the metric ρdefined by

$\rho(f_1, f_2) = sup\ \{|f_1(x) - f_2(x)|: x \in X||||x|||\leq 1\}, f_1, f_2 \in X^*$.

Proof: clearly $\rho$is a metric on$X$.

To prove condition (i) of snals let $f_1, f_2 \in X^*$and$x \in B_X$.

Then $|f_1(x)| \leq |f_1(x) - f_2(x)| + |f_2(x)| \leq \rho(f_1, f_2) + |||f_2|||$.

Since $x \in B_X$was arbitrary, if follows $|||f_1||| \leq \rho(f_1, f_2) + |||f_2|||$.

Similarly $|||f_2||| \leq \rho(f_1, f_2) + |||f_1|||$.Hence it follows $|\, |||f_1||| - |||f_2|||\, | \leq \rho(f_1, f_2)$

Now let $x \in B_X$. By the definition of almost linear functional we have that

$f_1(x) - f_2(x) \leq f_1(x) + f_2(-x) = f_1(x) + (-lof_2)(x) \leq |||f_1 + (-1of_2)|||$

Similarly $f_2(x) - f_1(x) \leq |||f_1 + (-1of_2)|||$

Hence for each $x \in B_X$we have$|f_1(x) - f_2(x)| \leq |||f_1 + (-1of_2)|||$

Therefore it follows that $\rho(f_1, f_2) \leq |||f_1 + (-1of_2)|||$

Hence condition (i) of snalsfollows.

To provecondition (ii) of snals let $f_1 \in X^*, i = 1,2,3$.

Then$\rho(f_1 + f_3,\ f_2 + f_3) = \sup\{|(f_1 + f_3)(x) - (f_2 + f_3)\,(x)|: x \in B_X\} = \rho(f_1,\ f_2)$

To provecondition (iii) of snals we show that for each $f \in X^*$, the function$\alpha \to \rho(\alpha o f, f)$is continuous at any $\alpha > 0$.

Indeed, for $\alpha > 0$we have $\rho(\alpha o f, f) = sup\{|f(\alpha x) - f(x)|: x \in B_X\} = |\alpha - 1|\ |||\ f\ |||$.

Hence X^* is a strong normed almost linear space.■

Lemma3.3: For any nals$X$, V_{X^*} is a Banach space.**Proof:**$V_{X^*}$is a normed linear space for the norm defined as in the definition of an almost linear functional. By Lemma 2.9 each$f \in V_{X^*}$is linear on X.We know that the dual space of a normed linear space is complete. Since $V_{X^*}$is a normed linear space it is also complete. Hence $V_{X^*}$is a Banach space.■

Lemma3.4: Let X be a nals with a basis B. Then for each $b_0 \in B \backslash V_X$there exists $f \in X^{\#}$ such that $f(b_0) = 1$ and $f(b) = 0$ for each$b \in B \backslash \{b_0\}$.If $b_0 \in W_X$, then $f \in X^*$.

Proof: Let $x \in X\backslash\{0\}$. Then $x = \sum_{i=1}^{n} \alpha_i b_i$, where $b_i \neq b_j$ for $i \neq j$, and $\alpha_i > 0$ for $b_i \in B\backslash V_X$.

Define $f(x) = 0$ if $b_0 \notin \{b_1, \dots, b_n\}$ and $f(x) = \alpha_{i_0}$ if $b_{i_0} = b_0$ for some $i_o \in \{1, \dots \dots, n\}$.

Define also $f(0) = 0$. Then f satisfies all conditions of an almost linear functional. Therefore $f \in X^{\#}$. Suppose now that $b_0 \in W_X$. By Lemma 2.10, we can suppose |||b_0|||=1.

Let $x = \alpha_0 b_0 + \sum_{i=1}^{k} \alpha_i b_i$, where $\alpha_0 > 0$, $b_i \neq b_j$ for $i \neq j$ such that $f(x) > 0$. Then by Lemma 2.13, we have $f(x) = \alpha_0 = ||| \alpha_0 b_0 ||| \leq ||| x |||$ and so $f \in X^*$, $||| f |||=1$.■

Theorem3.5: Let X be a nals such that W_X has a basis, then $X^* \neq \{0\}$.

Proof: Since W_X has a basis, by Lemma 3.6 there exists $f \in (W_X)^*\backslash\{0\}$.
Let $x \in X$ and define $f_1(x) = f(x - x)$. Then $f_1 \in X^{\#}$, $f_1 \neq 0$ and for each $x \in X$, we have that $0 \leq f_1(x) \leq |||f|||\, |||x - x||| \leq 2\, |||f|||\, |||x|||$

This implies $||| f_1 ||| < \infty$. Hence $f_1 \in X^*\backslash\{0\}$. Thus $X^* \neq \{0\}$.■

Corollary3.6: If the nals X has a basis, then $X^* \neq \{0\}$■

Theorem3.7: Let X be a nals with a basis B such that $card(B\backslash V_X) < \infty$, then $X^* = \{f \in X^{\#}: f/V_X \in (V_X)^*\}$.

Proof: Let $f \in X^{\#}$, $f/V_X \in (V_X)^*$. If $f \notin X^*$, then there exists $x_n \in x$, $|||x_n||| \leq 1$, $n \in N$, such that $|f(x_n)| \to \infty$. Let $B\backslash V_X = \{b_1, \dots . b_k\}$. Then we have that $x_n = \sum_{i=1}^{k} \alpha_{n_i} b_i + v_n, \alpha_{n_i} \geq 0$, $v_n \in V_X, n \in N$.

Now the sequence $\{\alpha_{n_i}\}_{n=1}^{\infty}, 1 \leq i \leq k$ are all bounded.

Since $|f(x_n)| = \sum_{i=1}^{k} \alpha_{n_i} f(b_i) + f(v_n)| \to \infty$, it follows that $|f(v_n)| \to \infty$.

Since $f/V_X \in (V_X)^*$, we must have $|||v_n||| \to \infty$.

On the other hand $|||v_n||| \leq |||x_n||| + ||| \sum_{i=1}^{k} \alpha_{n_i} b_i|||$ for each $n \in N$.

It is a contradiction since the right hand inequality is bounded. Hence $f \in X^*$.■

Corollary3.8: If the nals$X$ has a basis B such that $card\ B < \infty$,then $X^\# = X^*$.

Proof: By the Theorem 3.7, we have$X^* = \{f \in X^\#: f/V_X \in (V_X)^*\}$, since X has a basis such that $card\ B < \infty$.So we must have that $X^\# = X^*$.■

Theorem3.9: Let X be a nals and let $f \in (W_X)^*$. Then there exists $f_1 \in X^*$such that$f_1/W_X = f$, $|||f_1||| = |||f|||$and $f_1/V_X = 0$.

Proof: Let X be a nals and let $f \in (W_X)^*$.

Define a function $f_1$by $f_1(x) = f(x - x)/2,\ x \in X$.

Then f_1 satisfies all the conditions of an almost linear functional.Hence $f_1 \in X^*$.

$$|||f_1||| = sup\{|f_1(x)|: |||x||| \leq 1\} = \sup\{|\frac{f(x-x)}{2}| : |||x||| \leq 1\}$$

$$= \sup\{|\frac{f(x)+f(-x)}{2}| : |||x||| \leq 1\} \leq \frac{1}{2}\sup\{|f(x)| + |f(-x)| : |||x||| \leq 1\}$$

$$= \frac{1}{2}\sup\{2|f(x)| : |||x||| \leq 1\} = \sup\{|f(x)| : |||x||| \leq 1\} = |||f||| < \infty.$$

Hence $|||f_1||| \leq |||f|||$

Let$w \in W_X$, then $f_1(w) = \frac{f(w-w)}{2} = \frac{f(w+(-w))}{2} = f(w)$

Hence $f_1(w) = f(w)$ for every $w \in W_X$.

Therefore $f_1/W_X = f$and we have$|||f_1||| \geq |||f|||$.This implies $|||f_1||| = |||f|||$.

Let $x \in V_X$, then $f_1(x) = \frac{f(x-x)}{2} = \frac{f(0)}{2} = 0$.for every $x \in V_X$.Hence $f_1/V_X = 0$.■

Let X be a nals. If $(W_X)^* \neq \{0\}$, then $X^* \neq \{0\}$.■ **3.1**

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Theorem3.10: The following assertions are equivalent.

i) There exists a nals$X$ such that $X^* = \{0\}$.

ii) There exists a nals X such that $X^* \neq \{0\}$, and $X^* = V_{X^*}$. That is X^* is a Banach space.

Proof : To prove (i) ⇒ (ii)

Suppose X is a nals such that $X^* = \{0\}$.

Let $Y = \{(x, \alpha): x \in X, \alpha \in R\}$and Let $s: Y \text{ x } Y \to Y$and $m: R \text{ x } Y \to Y$be defined by

$s((x_1\ \alpha_1), (x_2, \alpha_2)) = (x_1{+}x_2, \alpha_1{+}\alpha_2)$ and $m\big(\gamma(x, \alpha)\big) = (\gamma x, \gamma\alpha)$

Let $0 \in Y$be the element (0,0). Then Y is an als and we have $V_Y = \{(v, \alpha): v \in V_X, \alpha \in R\}$and $W_Y = \{(w, 0): w \in W_X\}$. Since $X \neq V_X$, then $Y \neq V_Y$. Define a norm on Y by $|||(x, \alpha)|||_1 = |||x||| + |\alpha|$.Then Y together with$|||\ .\ |||_1$is a nals.

Clearly the function f_0 defined on $Y$by $f_0\big((x, \alpha)\big) = \alpha, (x, \alpha) \in Y$belongs to$V_{Y^*}$**,** and $|||\ f_0\ |||_1$=1.We show that $Y^* = V_{Y^*}$.Let $f \in Y^* \backslash V_{Y^*}$By Lemma 2.8,there exists $(W_0, 0) \in W_Y, W_0 \in W_X$such that $f\big((w, 0)\big) > 0$.Define the functional $f_1$on X by $f_1(x) = f\big((x, 0)\big), x \in X$.

Then $f_1 \in X^*$and by (i) $f_1 = 0$, a contradiction Since $f_1(W_0) = f((W_0, 0)) > 0$.

Therefore$V_{Y^*} = Y^*$.To show$(ii) \Longrightarrow (i)$.

Let X be a nals such that $X^* = V_{X^*} \neq \{0\}$.

Since X is not a linear space that $W_X \neq \{0\}$and we have $(W_X)^* = \{0\}$.Thus there exists nalsW_X such that $(W_X)^* = \{0\}$.■

Theorem3.11: Let X be a nals with a basis B.

i) For each $f \in (V_X)^{\#}$there exists $f_1 \in V_{X^{\#}}, f_1/V_X = f$.

ii) If $card(B/V_X) < \infty$, then for each $f \in (V_X)^*$,there exists $f_1 \in V_{X^*}$such that $f_1/V_X = f$.

Proof: Suppose B has the property that for each $b \in B/V_X$we have $-b \in B/V_X$.

i) Let $f \in (V_X)^{\#}\backslash\{0\}$and let $x \in X/V_X$.Now there exists unique $b_1, \dots, b_n \in B\backslash V_X, \alpha_i > 0, 1 \le i \le n$ and $v \in V_X$such that $x = \sum_{i=1}^{n} \alpha_i b_i + v$**3.2**Define $f_1(x) = f(v)$and for $v \in V_X$define $f_1(v) = f(v)$.Then clearly $f_1 \in X^{\#}$and f_1 is an extension of f. To show that $f_1 \in V_{X^{\#}}$, by Lemma 2.8, we must show that $f_1(-x) = -f_1(x)$ for each $x \in X\backslash V_X$.

If X has the representation given in (3.2), then $-x = \sum_{i=1}^{n} \alpha_i(-b_i) - v$and so $f_1(x) = f(-v) = -f(v) = f_1(x)$.

ii). Suppose$card\ (B\backslash V_X) < \infty$. Let $f \in (V_X)^{*}\backslash\{0\}$.Then by (i) there exists $f_1 \in V_{X^{\#}}$. $f_1/V_X = f$.Then by Theorem 3.7, the result follows.■

Corollary3.12: Let X be a nals with a basis B, such that $card\ (B\backslash V_X) < \infty$. Then $X^{*}$is total over X.

Proof: Suppose $B\backslash V_X = \{ b_1, \dots, b_n\}$ Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ for each $f \in X^{*}$. Now we have that$x_i = \sum_{j=1}^{n} \alpha_{ij} b_j + v_i, \alpha_{ij} \ge 0, 1 \le j \le n, v_i \in V_X, i = 1,2$.

By Lemma 3.6 for each $b_j \in B\backslash V_X$ there exists $f_j \in X^{\#}$ such that $f_j(b_j) = 1$and $f_j(b) = 0$for $b \in B\backslash\{b_j\}$.By Theorem 3.9, $f_j \in X^{\#}$, hence by our assumption,

$$f_j(x_1) = f_j\left(\sum_{j=1}^{n} \alpha_{1j}\ b_j\right) + f_j(v_1) = \sum_{j=1}^{n} \alpha_{1j}\ f_j(b_j) + f_j(v_1) = \alpha_{1j}. \quad f_j(x_2) = f_j\left(\sum_{j=1}^{n} \alpha_{2j}\ b_j\right) + f_j(v_2) = \sum_{j=1}^{n} \alpha_{2j}\ f_j(b_j) + f_j(v_2) = \alpha_{2j}.$$

Hence $\alpha_{1j} = \alpha_{2j}$for $1 \le j \le n$.Consequently for each $f \in X^{*}$, we get $f(v_1) = f(v_2)$.Since $V_X$is a normed linear space, by Theorem 3.11, it follows that $v_1 = v_2$. Therefore $x_1 = x_2$and hence $X^{*}$is total over X.■

Theorem3.13: Let X be a nals such that$X = W_X + V_X$.Then for each $f \in (V_X)^{*}$there exists a norm preserving extension $f_1 \in V_{X^{*}}$.

Proof: Let $f \in (V_X)^{*}\{0\}$.Now for each $x \in X$, there exists unique $w \in W_X$ and $v \in V_X$such that $x = w + v$.Define $f_1(x) = f(v)$. Clearly $f_1 \in X^{\#}$and $f_1 \in V_{X^{\#}}$.Now we get $|f_1(x)| = |f(v)| \le ||| f ||| \, ||| v ||| \le ||| f ||| \, ||| x |||$and so $||| f_1 ||| = ||| f |||$.■

Theorem3.14: Let$X$ be snals such thatρ is a metric and if $x \in X\backslash(W_X + V_X)$. Suppose $X = \{\alpha x_0 + \mu(-x_0) + w + v: w \in W_X, v \in V_X, \alpha, \mu \geq 0\}$then i) for each $f \in (V_X)^*$ there exists $f_1 \in V^*$ such that $f_1\backslash V_X = f$ ii) $V_{X^*} \neq \{0\}$ and iii) for each $f \in (W_X + V_X)^*$there exists $f_1 \in X^*$ such that $f_1\backslash(W_X + V_X) = f$.

Proof: We first show that $X = X_1 \cup X_2 \cup (W_X + V_X)$**3.3** Where $X_1 = \{\alpha x_0 + w + v: \alpha > 0, w \in$ *WX,v∈VX}*

$X_2 = \{-\alpha x_0 + w + v: \alpha > 0, w \in W_X, v \in V_X\}$and we have that $X_1 \cap X_2 = \emptyset$, $X_i \cap (W_X + V_X) = \emptyset, i = 1,2$.

Since $X_1 \cup X_2 \cup (W_X + V_X) \subset X$ is obvious, let $x \in X$.

Say $x = \alpha x_0 + \mu(-x_0) + w + v, \alpha, \mu \geq 0, w \in W_X, v \in V_X$.

If $\alpha = \mu$ then, since $\alpha(x_0 - x_0) \in W_X$, it follows that $x \in W_X + V_X$. If $\alpha > \mu$then$x = (\alpha - \mu)x_0 + \mu(x_0 - x_0) + w + v \in X_1$.
Similarly if $\alpha < \mu$then $x \in X_2$. This proves (3.3). Since $\pm x_0 \notin W_X + V_X$, it follows that $X_i \cap (W_X + V_X) = \emptyset, i = 1,2$. Let $x \in X_1 \cap X_2$. Then there exists $\alpha_i > 0$, $w_i \in W_X$, $v_i \in V_X, i = 1,2$,such that $x = \alpha_1 x_0 + w_1 + v_1 = -\alpha_2 x_0 + w_2 + v_2$.

Hence $(\alpha_1 + \alpha_2)x_0 + w_1 + v_1 = \alpha_2(x_0 - x_0) + w_2 + v_2 \in W_X + V_X$.

Now againit follows that $(\alpha_1 + \alpha_2)x_0 \in W_X + V_X$.

It is a contradiction since $\alpha_1 + \alpha_2 > 0$and $x_0 \notin W_X + V_X$.
Therefore $X_1 \cap X_2 = \emptyset$.For $Y = W_X + V_X$, we get that any $x \in X$can be uniquely represented in the form$x = \alpha x_0 + w + v, (\alpha \in R, w \in W_X, v_i \in V_X)$ **3.4**
i). Let $f \in (V_X)^*\backslash\{0\}$.If $x \in X$has the representation given by (3.4) define $f_1(x) = f(v)$.

Clearly $f_1 \in V_{X^\#}$.If $f_1 \notin V_{X^*}$then there exists $x_n \in X$, $|||x_n||| \leq 1, n \in N$, such that $|f_1(x_n)| \to \infty$.

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Suppose $x_n = \alpha_n x_0 + w_n + v_n, \alpha_n \in R, w_n \in W_X, v_n \in V_X, n \in N$. Suppose that for an infinity of n we have $\alpha_n \geq 0$and without loss of generality we can suppose $\alpha_n \geq 0$for all $n \in N$. Now it follows that $|||\alpha_n x_0 + v_n||| \leq |||x_n||| \leq 1$ for each $n \in N$. And so the sequence $\{\alpha_n\}_{n=1}^{\infty}$is bounded. Then $|||v_n||| \leq 1+\alpha_n|||x_n|||, n \in N$hence the sequence $\{|||v_n|||\}_{n=1}^{\infty}$is bounded. We get the same conclusion if $\alpha_n \leq 0, n \in N$.

Then we work with $-x_0$ instead of x_0. Now since $|f_1(x_n)| = |f(v_n)| \to \infty$ and$f \in (V_X)^*$We obtain that$v_n \to \infty$a contradiction. Hence $f_1 \in V_{X^*}$.

ii) If $V_X \neq \{0\}$, then by (i) we get $V_{X^*} \neq \{0\}$. Suppose now $V_X = \{0\}$.

Let $x \in X$, then by (3.3) there exists unique $\alpha \in R$,$w \in W_X$such that $x = \alpha x_0 + w$. Define $f(x) = \alpha|||x_0|||$. Clearly we have $f \in V_{X^{\#}}$. Now we get $f(x) = \alpha|||x_0||| \leq |||\alpha x_0 + w||| = |||x|||$Hence $f \in V_{X^*}\setminus\{0\}$.

iii). Let$f \in (W_X + V_X)^*\setminus\{0\}$. If $V_X = \{0\}$then the result follows by Theorem 3.9. Suppose now $V_X \neq \{0\}$ By (i) there exists $f_2 \in X^*$ such that $f_2/V_X = f/V_X$and $f_2/W_X = 0$. By Theorem 3.9, there exists $f_3 \in X^*$such that $f_3/W_X = f/W_X$and $f_3/V_X = 0$.

Let $f_1 = f_2+f_3$. Then $f_1 \in X^*$and we have $f_1/(W_X + V_X) = f$.■

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