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Some Coefficients Method Of Solving Riccati Equation By Lie Group Symmetry

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Abstract

On solving Riccati Equation [1] by symmetry groups of the form $\frac{dy}{dx}= \boldsymbol{R}(\boldsymbol{x})\boldsymbol{y}^{2}+\boldsymbol{Q}(\boldsymbol{x})\boldsymbol{y}+\boldsymbol{P}(\boldsymbol{x})$. in practice finding the solutions of it , is usually a much more difficult problem solving the Riccati Equation by inspired guesswork, or geometric intuition, the steps used when solving all first order differential equations involve some assumptions and guesses of the form of symmetry for a given differential equation. it is possible to ascertain a particular solution by linearized symmetry condition[2]for example but these methods take long time for finding parameters Lie group and hard work to make Simplifying bit and comparing coefficients of powers of equations.

Solutions of the Riccati equation with coefficients $\frac{dy}{dx}= \boldsymbol{R}(\boldsymbol{x})\boldsymbol{y}^{2}+\boldsymbol{Q}(\boldsymbol{x})\boldsymbol{y}+\boldsymbol{P}(\boldsymbol{x})$.

Are presented. The solutions are obtained by assuming certain relations among the coefficients R(x), Q(x) and P(x) of the Riccati equation, we obtain symmetry cases for the Riccati equation. For each case the general solution of the Riccati equation is also presented.

Of Of the form $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}})=(\boldsymbol{e}^{\boldsymbol{\lambda}\times \boldsymbol{coefficient}(y^{2})}\,\boldsymbol{x},\boldsymbol{e}^{\boldsymbol{\lambda}\times \boldsymbol{coefficient}(y)}\boldsymbol{y})$

or

$$(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}})=(\boldsymbol{e}^{\boldsymbol{\lambda}\times \boldsymbol{coefficient}(y^{2})}\,\boldsymbol{x},\boldsymbol{e}^{-\boldsymbol{\lambda}\times \boldsymbol{coefficient}(y)}\boldsymbol{y})$$

can be used to find some common Lie symmetries (for some parameter) scaling $(\hat{x},\hat{y})$ But this method, do not always give the required tangent form, and satisfy symmetry condition too**.**

The main aim to provide a clear solution and clear computation by mat lab by looking for possibility of actually finding *Lie Point Symmetries* as given above using the same (and more) equations for solving Riccati Equation Although Lie group symmetry has been widely employed for solving many Riccati equations.

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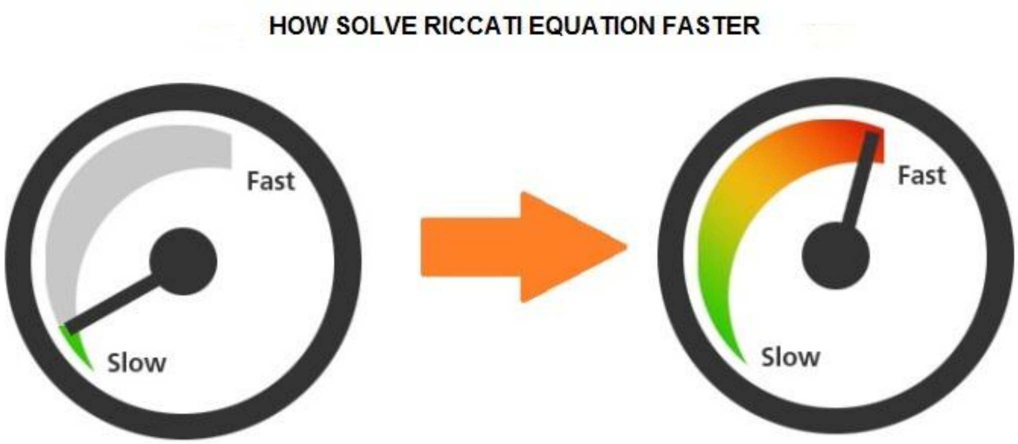

1.0 Introduction

Differential equations are used to model various phenomenon's in our world, from the unfold of infectious diseases to the behavior of periodic event waves. Naturally, the study of differential equations plays a significant role within the physical sciences. These equations are typically non-linear and resolution them needs distinctive and artistic strategies.

Sophus Lie [3] (pronounced Lee), a Norwegian scientist born in 1842, developed these techniques. He was the primary to find the link between group theory and ancient methods for finding answer curves.

His ground breaking discovery concerned mistreatment groups of purpose transformations to solve differential equations the study of symmetry provides one among the foremost appealing sensible applications of group theory. He investigated the continuous groups of transformations leaving differential equations invariant, making what's currently known as the symmetry analysis of differential equations. His aim was to resolve non-linear differential equations, that to some extend are could also be cumbersome to resolve.

2.0 Literature Review

As explained in [4], a Lie group could be a mixing of the algebraically thought of a group and therefore the differential-geometric thought of a manifold.

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The distinctive feature of a Lie group from the lot of general kinds of groups is that it additionally carries the structure of a smooth manifold, and therefore the group elements are often continuously varied. Basically, this mixture of algebra and calculus results in a strong techniques for the study of symmetry.

Definition: Lie group

This is a group is that is also a smooth manifold and such that the multiplication map

$$\mu : G \times G \to G \qquad \mu : (g, h) \mapsto \qquad g, h \in G$$

and the inversion map

$$i : G \to G \qquad i : g \mapsto g \qquad g \in G$$

are smooth maps between manifolds.

An r-parameter Lie group carries the structure of an r-dimensional manifold. A Lie group can also be considered as a topological group (i.e., a group endowed with a topology with respect to which the group operations are continuous) that is also a manifold.

2.2 Lie group symmetries

As it is defined in [5], Lie group could be a group of symmetries with a parameter λ ϵ R.

Lie group symmetries are functions from R2 to R2. Let A be a set of points (x, y) ϵ R2 and let B be a set of points (xˆ,"ŷ") ϵ R2. A Lie group P_λ maps A to B:

$$P_\lambda : A \longmapsto B.$$

Here, xˆ could be a function of x, y, and λ and "ŷ" is additionally a function of x, y, and λ. Therefore, a Lie group also can be written as $P_\lambda : (x, y) \longmapsto (f(x, y, \lambda))$.

Lie groups should meet the subsequent restrictions:

1) P_λ is one-to-one and onto

2) $P_{\lambda 2} \circ P_{\lambda 1} = P_{\lambda 2 + \lambda 1}$

3) $P_0 = I$

4) $\forall \lambda_1 \in R, \exists \lambda_2 = -\lambda_1$ such that $P_{\lambda 2} \circ P_{\lambda 1} = P_0 = I$

Lie groups **May not** essentially be defined over the whole plane. We are going to be dealing with *local groups*. The group action of a *local group* is not necessarily defined over the entire real number plane . Consider the next example.

The following Lie group is only defined if $\lambda < \frac{1}{x}$ when $x > 0$ and $\lambda > \frac{1}{x}$ when $x < 0$:

$$P_\lambda : (x, y) \mapsto$$

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$$(\hat{x},\hat{y}) = \left(\frac{x}{1-\lambda x}, \frac{y}{1-\lambda x}\right).$$

We can see at a glance that if $\lambda < \frac{1}{x}$, P_λ is undefined. We can verify that the identity For P_λ is $\lambda = 0$:

$$P_0: (x,y) \longmapsto (\hat{x},\hat{y}) = \left(\frac{x}{1}, \frac{y}{1}\right) = (x,y).$$

Therefore, the interval on which P_λ is defined should include the origin. If $x > 0$ and $\lambda > \frac{1}{x}$, then the identity is not included during this interval. Similarly, if $x < 0$ and $\lambda < \frac{1}{x}$ then the origin is not included. Therefore, in order for the group to have an identity, it can only be defined when $\lambda < \frac{1}{x}$, for $x > 0$ and $\lambda > \frac{1}{x}$ for $x < 0$. This means that, for a fixed λ, the domain of P_λ is $\frac{1}{-\lambda} < x < \frac{1}{\lambda}$ and $y \in R$. The range is $\frac{1}{-2\lambda} < \hat{x}$ and $\hat{y} \in R$.

2.3 *Find Lie Point Symmetries of* Riccati Equation

In this section is centered on Method Of finding Lie group Symmetry Riccati Equation $\frac{dy}{dx} = R(x)y^2 + Q(x)y + P(x)$.

Here, we will use form below

$$(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}) = (\boldsymbol{e}^{\lambda\times \mathrm{coeffs}(y^2)}\,\boldsymbol{x}, \boldsymbol{e}^{\lambda\times \mathrm{coeffs}(y)}\boldsymbol{y})$$

can be used to find some common Lie symmetries (for some parameter) including translations, scaling and rotations.

Example 1. In solving *of* Riccati Equation $\frac{dy}{dx} = \frac{3+3y^2}{x}, x \neq 0$. to determine the *Lie Point* symmetries of a Riccati Equation and we are going to show that it's a symmetry for every fixed. It satisfies all of the properties of Lie symmetry groups

Now we can rewrite equation as

$$\frac{dy}{dx} = \frac{1+y^2}{x}, x \neq 0.$$

By division 3

$\mathrm{coeffs}(y^2) = 1$, $\mathrm{coeffs}(y) = 0$

Substituting this into formula ,

$$(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}) = (\boldsymbol{e}^{\lambda\times \mathrm{coeffs}(y^2)}\,\boldsymbol{x}, \boldsymbol{e}^{\lambda\times \mathrm{coeffs}(y)}\boldsymbol{y})$$

$$((\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}) = (\boldsymbol{e}^{\lambda\times 1}\,\boldsymbol{x}, \boldsymbol{e}^{\lambda\times 0}\boldsymbol{y})$$

we would like to get:

It has the following symmetry:

$$(\hat{x},\hat{y}) = (e^{\lambda}x, y).$$

We are going to show that it's a symmetry for every fixed. It satisfies all of the properties of Lie symmetry groups Equation in **Example 1**

$$P_\lambda : (x,y) \longmapsto (\hat{x},\hat{y})(e^{\lambda}x, y).$$

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1) the mapping of this equation is one-to-one and onto.

This means that if 2 points (x_1, y_1) and (x_2, y_2) map to similar $(\hat{x}, \hat{y})$ then $(x_1, y_1) = (x_2, y_2)$ and for each $(\hat{x}, \hat{y})$ on the interval that Equation of example 1 is defined (in this case,

The entirety of R^2), there is a point (x, y) that maps to $(\hat{x}, \hat{y})$. Consider the mapping Equation for a fixed λ we are able clearly see that it is each one-to-one and onto for $\hat{y}$ because $\hat{y} = y$. This can be the identity mapping. Now consider $\hat{x} = e^{\lambda}x$. If we have

$e^{\lambda} x_1 = e^{\lambda} x_1 = \hat{x}$, we are able to divide by e^{λ} to check that $x_1 = x_2$. To indicate that there is a x that corresponds with each $\hat{x}$ in R, we can divide $\hat{x}$ by e^{λ}. Then $x = e^{-\lambda}\hat{x}$ Therefore, the mapping Equation is one-to-one and onto.

2) The group satisfies the composition property of Lie groups. Suppose we tend to take $P_{\lambda 2} \circ P_{\lambda 1}$. Apply $P_{\lambda 1}$ to the point (x, y)

To get

$$(\hat{x}_1, \hat{y}_1) = (e^{\lambda}x, y)$$

Then apply $P_{\lambda 2}$ to $(e^{\lambda}x, y)$ to get

$$(\hat{x}_2, \hat{y}_2) = (e^{\lambda_2}e^{\lambda_1}x, y) = (e^{\lambda_2+\lambda_1}x, y).$$

This is what we tend to get after we apply $P_{\lambda_2+\lambda_1}$, and therefore the composition property is satisfied.

3) P_0 is that the identity:

$$(\hat{x}, \hat{y}) = (e^0 x, y) = (x, y).$$

4) For all $\lambda_1 \in R$ there is an inverse.

When we apply $P_{-\lambda_1} \circ P_{\lambda_1}$, we get

$$(\hat{x}, \hat{y}) = (e^{-\lambda_1}$$

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$(e^{\lambda_1} x), y) = (e^{\lambda_1 - \lambda_1} x, y) = (x, y).$

Therefore, all of the Lie group properties are satisfied.

2.4 The Symmetry Condition

In general, we are working with differential equations of the form:

$$\frac{dy}{dx} = \omega(x, y).$$

In order to satisfy the symmetry condition, the point $(\hat{x}, \hat{y})$ should even be on a solution curve to the differential equation above :

$$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}).$$

And the transformation $(x, y) \leftrightarrow (\hat{x}, \hat{y})$ is asymmetry of equation above if and only if the symmetry condition is satisfied

$$\frac{\hat{y}_x + \omega(x,y)\hat{y}_y}{\hat{x}_x + \omega(x,y)\hat{x}_y} = \omega(\hat{x}, \hat{y})$$

In example 1 we will verify with the symmetry condition the *Lie Point* symmetries $(\hat{x}, \hat{y}) = (e^{\lambda} x, y)$
First we will calculate:

$$\hat{y}_x = 0, \hat{y}_y = 1$$

And

$$\hat{x}_x = e^{\lambda}, \hat{x}_y = 0.$$

Evaluating the left side of Equation (2.9) we get:

$$\frac{d\hat{y}}{d\hat{x}} = \frac{\frac{3+3y^2}{x}}{e^{\lambda}} = \frac{3+3y^2}{e^{\lambda} x}.$$

Evaluating the right side of Equation (2.9), we get:

$$\frac{dy}{dx} = \frac{3 + 3\hat{y}^2}{\hat{x}} = \frac{3 + 3y^2}{e^{\lambda} x}.$$

Thus the symmetry condition is satisfied for the symmetry $(\hat{x}, \hat{y}) = (e^{\lambda} x, y)$ of the Riccati Equation.

3.0 solve Riccati Equation

The various steps to be used when solving Riccati Equation will be presented here. If the symmetry form as found, then the Riccati Equation can be solved with ease. Here, it is shown and proven that one does not really need to guess the form of symmetry

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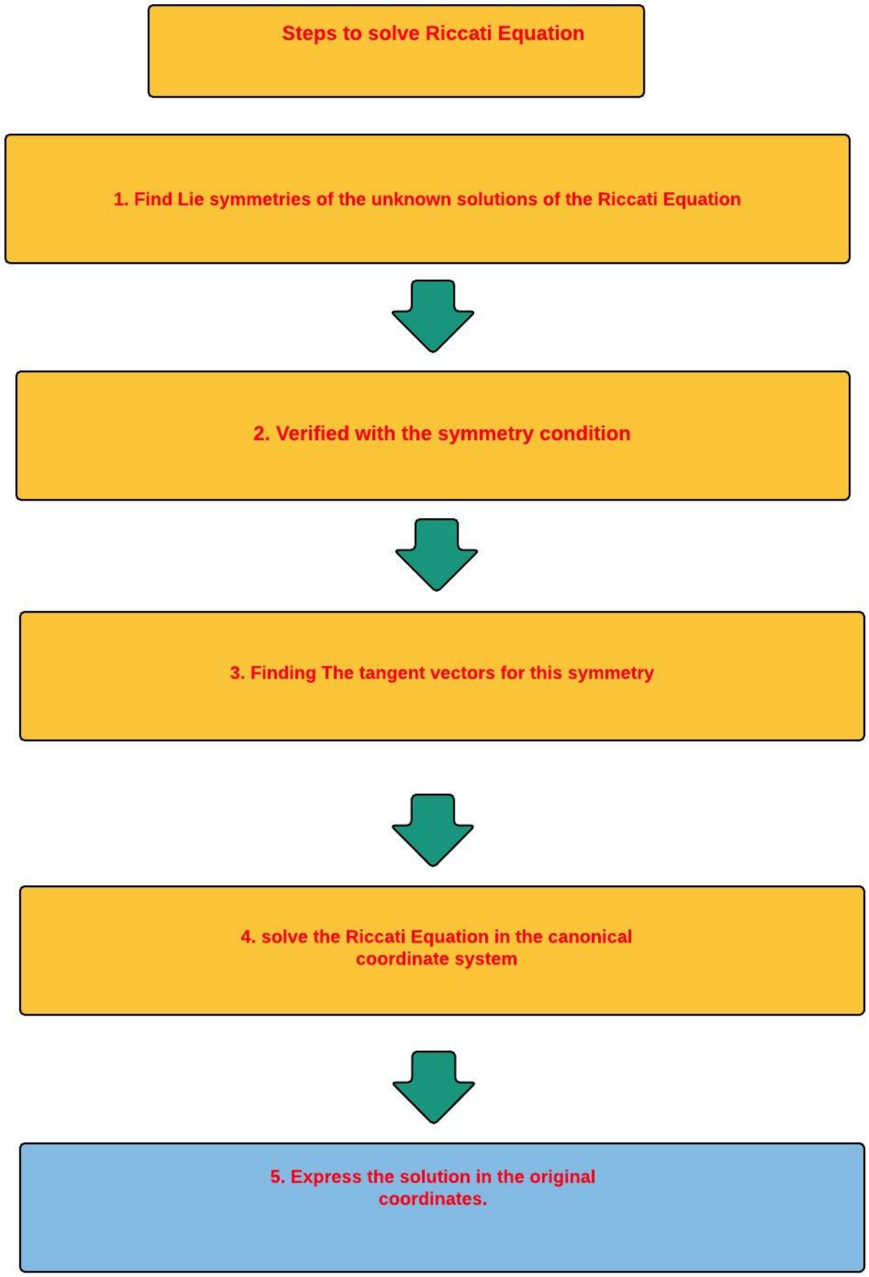

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3.1 Steps to solve Riccati Equation

1. Find Lie symmetries of the unknown solutions of the Riccati Equation, say $(\hat{x},\hat{y}) = \left(e^{\lambda\times\text{coeffs}(y^2)}x, e^{\lambda\times\text{coeffs}(y)}y\right)$

2. Verified with the symmetry condition.

3. *Finding* The tangent vectors for this symmetry

4. Substitute the canonical coordinates into

$$\frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y}$$

And solve the Riccati Equation in the canonical coordinate system.

5. Express the solution in the original coordinates.

Example 2: In solving the Riccati Equation $\frac{dy}{dx} = 2y^2 - \frac{2y}{x}, x \neq 0.$, the calculated associated Point Symmetries$(\hat{x},\hat{y}) = (e^{\lambda}x, e^{-\lambda}y)$. Satisfied the symmetry condition

The canonical coordinates are found by solving $\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)} = \frac{-y}{x}$ to get r and s.

Families of functions that remain constant for r are sought here, so $r = c = xy$.

The second coordinate s is found by integrating to get $= \int \frac{dx}{x} = \ln x$..

The next step is to express the differential equation in the canonical coordinates by computing

$$\frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y}$$

Proper substitution gives;

$$= \frac{\frac{1}{x}}{y + (2y^2 - \frac{2y}{x})x}$$

$$= \frac{1}{2x^2y^2 - yx}.$$

Then substitute in $x = e^s$ and$y = re^{-s}$:

$$\frac{ds}{dr} = \frac{1}{2e^{2s}r^2e^{-2s} - re^{-s}e^{2s}}$$

$$= \frac{1}{r^2 - r}.$$

Then we are able to integrate. The equation

$$\frac{ds}{dr} = \frac{1}{r^2 - r}$$

Is separable:

$$ds = \frac{dr}{r^2 - r}$$

And therefore

$$s = \int \frac{dr}{r^2 - r}.$$

We can integrate this then:

$$s = \ln\left(\left|\frac{1}{r} - 1\right|\right) + c$$

Returning to the original coordinates

$$y = -\frac{e^c}{x(e^c - x)}.$$

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Example 3: In solving the Riccati Equation $\frac{dy}{dx} = 3xy^2 - \frac{6y}{x}, x \neq 0..$, the calculated associated Lie Point Symmetries $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = (\boldsymbol{e^{\lambda}x}, \boldsymbol{e^{-2\lambda}y})$. Satisfied the symmetry condition

The tangent vectors for this symmetry are form$\xi(x,y) = x, \eta(x,y) = -2y$.

The canonical coordinates are found by solving $\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)} = \frac{-2y}{x}$ to get r and s.
Families of functions that remain constant for r are sought here, so $r = c = x^2y$.
.

The second coordinate s is found by integrating to get$s = \int \frac{dx}{x} = \ln x..$
The next step is to express the Riccati Equation in the canonical coordinates by computing

$$\boldsymbol{\frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y}}$$

Proper substitution gives;

$$\frac{ds}{dr} = \frac{\frac{1}{x}}{2xy + \frac{dy}{dx}x^2}$$
$$= \frac{\frac{1}{x}}{2xy + 3x^3y^2 - 6xy}$$
$$= \frac{1}{3x^4y^2 - 4x^2y}$$

Then substitute in $x = e^s$ and$y = re^{-2s}$:

$$\frac{ds}{dr} = \frac{1}{3e^{4s}r^2e^{-4s} - 4e^{2s}re^{-2s}}$$
$$= \frac{1}{3r^2 - 4r}.$$

Then we are able to integrate. The equation

$$\frac{ds}{dr} = \frac{1}{3r^2 - 4r}$$

Is separable:

$$ds = \frac{dr}{3r^2 - 4r}$$

And therefore

$$s = \int \frac{dr}{3r^2 - 4r}.$$

We can integrate this using partial fractions:

$$s = \frac{\ln(|\frac{4}{r} - 3|)}{4} + c$$

Returning to the original coordinates

$$y = -\frac{4e^{4c}}{x^2(x^4 - 3e^{4c})}$$

3.2 Mat lab for solving the Riccati Equation

It may be simple to calculate the symmetries of some Riccati Equation, to find the Lie Point Symmetries for solving the Riccati Equation using
Mat lab is an interpreted language for numerical computation. It allows one to perform numerical calculations, and visualize the results without the need for complicated and time consuming programming, by this method but before using it you must verify

with the symmetry condition of the *Lie Point* symmetries, Input Riccati equation in form: [f2(x)*y^2+f1(x)*y+f0(x)]

The table 6.2.1 below shows examples how get Lie Point Symmetries**:**

The table 3.21 below shows examples how get Lie Point Symmetries:

Equation	f2	f1	f0	Lie Point Symmetries
$\frac{7-7y^2}{x}$	7	0	7	$(\hat{x},\hat{y}) = (e^{\lambda}x, y)$
$\frac{5y^2+5}{x}$	5	0	5	$(\hat{x},\hat{y}) = (e^{\lambda}x, y)$
$9y^2 - \frac{9y}{x}$	9	9	0	$(\hat{x},\hat{y}) = (e^{\lambda}x, e^{-\lambda}y).$
$y^2 + \frac{y}{x}$	1	1	0	$(\hat{x},\hat{y}) = (e^{\lambda}x, e^{-\lambda}y).$
$\frac{dy}{dx} = 8xy^2 - \frac{16y}{x}$	8	16	0	$(\hat{x},\hat{y}) = (e^{\lambda}x, e^{-2\lambda}y).$
$\frac{dy}{dx} = 12xy^2 + \frac{6y}{x}$	12	6	0	$(\hat{x},\hat{y}) = (e^{2\lambda}x, e^{-\lambda}y).$

Table 3.2.1

4.0 Conclusion & Recommendations

It has been shown that the coefficients R(x), Q(x) and P(x) of the Riccati equation have Lie Point Symmetries it has been shown that one can actually find Lie Point Symmetries and the tangent vectors corresponding to the form of Lie symmetry of Riccati equation. In fact, the tangent vectors of differential equations have been computed for to get the same values exactly as given in [5], And show how Shortcut time for finding Lie Point Symmetries exactly the contrary inspired guesswork take long time for finding parameters Lie group. In this way we must apply the symmetry condition to complete the solution otherwise if Lie Point Symmetries not satisfied symmetry condition you will back to use other ways. But there are few relations between coefficients R(x), Q(x) and P(x) of the Riccati equation can be used for solving. It is not certain about how the symmetries of equations like these are achieved, or maybe they have forms of symmetries which are assumed

There are some cases where the search for the symmetries difficult or may fail. In addition, it's additionally not quite clear how we tend to get symmetries of Riccati equation that embrace a one-parameter Lie group of inversions.

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