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# Some Type of Infinite Series

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***Abstract***

*In this paper, we study some type of infinite series. Each value of this type of infinite series can be obtained by using a recursive formula. On the other hand, we obtain the necessary and sufficient condition regarding the value of the infinite series are positive integers. Moreover, a positive integer valued infinite series is proposed to demonstrate our results, and we use Maple to check the answers.*

***Keywords***

*Infinite series, recursive formula, positive integers, Maple*

## 1. Introduction

Infinite series are the most powerful and useful tools that we encountered in calculus and engineering mathematics courses. They are the major tools used in analyzing differential equations, in developing methods of numerical analysis, in defining new functions, in estimating the behavior of functions, and more. For the study on infinite series and their applications, we can refer to [1].

This article considers the following infinite series which is not easy to obtain their explicit expressions:

$$\sum_{n=1}^{\infty}\frac{n^p}{a^n}, \tag{1}$$

where $p$ is a non-negative integer, $a$ is a real number, and $a>1$. We can determine the recursive formula for this type of infinite series, i.e., Theorem 1. Moreover, using Theorem 1, we can easily obtain the necessary and sufficient condition for the values of the infinite series (1) are positive integers, i.e., Theorem 2, and hence generalize this result in [4], which shows that the infinite series $\sum_{n=1}^{\infty}\frac{q(n)}{2^n}$ are integers, where $q(x)$ is any polynomial of $x$ with integer coefficients.

## 2. Preliminaries

At first, we introduce some notations and theorems used in this paper.

### 2.1. Notations:

Suppose that $n,k$ are non-negative integers, and $n\ge k$. Define $\binom{n}{k}=\frac{n!}{k!(n-k)!}$.

Next, the following theorem can be found in [2, p167].

### 2.2. Ratio test:

For a given infinite series $\sum_{n=1}^{\infty}a_n$, we evaluate the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L$. There are three possibilities:

(1) if $L<1$, then the series converges absolutely,
(2) if $L>1$, then the series diverges,
(3) if $L=1$, then the test is inconclusive.

We also need the following theorem which appeared in [3, p59].

### 2.3. Binomial theorem:

Suppose that $x,y$ are real numbers, and $n$ is a positive integer, then

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k. \tag{2}$$

## 3. Main Results

By ratio test, the infinite series (1) converge for all $a>1$. In the following, we obtain the recursive formula of (1).

**Theorem 1** *Suppose that $p$ is a non-negative integer, $a$ is a real number, and $a>1$. Let $M_p=\sum_{n=1}^{\infty}\frac{n^p}{a^n}$, then the sequence $\{M_p\}_{p=0}^{\infty}$ satisfying the recursive formula*

$$M_p=\frac{a}{a-1}\sum_{k=1}^{p}(-1)^{k+1}\binom{p}{k}M_{p-k}, \tag{3}$$

*for all $p\ge1$, and $M_0=\frac{1}{a-1}$.*

**Proof** *Case (1).* If $p=0$, then

$$M_0=\sum_{n=1}^{\infty}\frac{1}{a^n}=\frac{1}{a-1}. \tag{4}$$

*Case (2).* If $p\ge1$, then using

$$M_p=\sum_{n=1}^{\infty}\frac{n^p}{a^n}, \tag{5}$$

we have

$$\frac{1}{a}M_p=\sum_{n=1}^{\infty}\frac{n^p}{a^{n+1}}. \tag{6}$$

It follows that

$$\left(1-\frac{1}{a}\right)M_p$$
$$=\sum_{n=1}^{\infty}\frac{[n^p-(n-1)^p]}{a^n}$$
$$=\sum_{n=1}^{\infty}\frac{\left[n^p-\sum_{k=0}^{p}\binom{p}{k}n^{p-k}(-1)^k\right]}{a^n}$$

(by binomial theorem)

$$=\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{p}\binom{p}{k}n^{p-k}(-1)^{k+1}}{a^n}.$$

Therefore,

$$M_p$$
$$=\frac{a}{a-1}\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{p}\binom{p}{k}n^{p-k}(-1)^{k+1}}{a^n}$$
$$=\frac{a}{a-1}\sum_{k=1}^{p}(-1)^{k+1}\binom{p}{k}M_{p-k}.$$

q.e.d.

On the other hand, using this recursive formula, we can determine the necessary and sufficient condition for the value of infinite series (1) are positive integers, and hence generalize the result in [4].

**Theorem 2** *If the assumptions are the same as Theorem 1, then* $\sum_{n=1}^{\infty}\frac{n^p}{a^n}$ *are positive integers for all non-negative* $p$ *if and only if* $a=\frac{m+1}{m}$ *for some positive integer* $m$.

**Proof** Only if part: if $\sum_{n=1}^{\infty}\frac{n^p}{a^n}$ are positive integers for all non-negative $p$, then by Eq. (4) we have $\frac{1}{a-1}$ is a positive integer. Assume that $\frac{1}{a-1}=m$ for some positive integer $m$, and hence $a=\frac{m+1}{m}$.

If part: if $a=\frac{m+1}{m}$ for some positive integer $m$, then $\frac{1}{a-1}$ is a positive integer, and hence $\sum_{n=1}^{\infty}\frac{1}{a^n}$ and $\frac{a}{a-1}$ are also positive integers. Using Eq. (3) and by mathematical induction yield $\sum_{n=1}^{\infty}\frac{n^p}{a^n}$ are positive integers for all non-negative $p$.

q.e.d.

## 4. Example

In the following, for the infinite series discussed in this article, we give an example to do calculation practically, and use Maple to verify our answers.

**4.1 Example:** If $a=\frac{5}{4}$, then

$$M_0=\sum_{n=1}^{\infty}\frac{1}{\left(\frac{5}{4}\right)^n}=4. \tag{7}$$

By Eq. (3) we obtain

$$M_1=5M_0=20, \tag{8}$$

$$M_2=5(2M_1-M_0)=180, \tag{9}$$

$$M_3=5(3M_2-3M_1+M_0)=2420, \tag{10}$$

$$M_4=5(4M_3-6M_2+4M_1-M_0)=43380. \tag{11}$$

Next, we use Maple to verify the correctness of Eqs. (7)-(11).

```
>sum(1/(5/4)^n,n=1..infinity);
                    4
>sum(n/(5/4)^n,n=1..infinity);
                    20
>sum(n^2/(5/4)^n,n=1..infinity);
                    180
>sum(n^3/(5/4)^n,n=1..infinity);
                    2420
>sum(n^4/(5/4)^n,n=1..infinity);
                    43380
```

## 5. Conclusion

In this paper, we mainly use binomial theorem to obtain a recursive formula of infinite series (1), and hence generalize the result in [4]. In fact, the applications of binomial theorem are extensive, it can be used to solve many difficult problems. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topics to other calculus and engineering mathematics problems and use Maple to demonstrate our results.

## References

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