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A study of nonlinear relativistic Hartree equation

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Abstract

The aim of this paper is to propose and analyze various numerical methods for some representative classes of nonlinear Hartree equations, which mainly arise in the problems of quantum mechanics and nonlinear optics. Extensive numerical results are also reported, which are geared towards demonstrating the efficiency and accuracy of the methods, as well as illustrating the numerical analysis and applications. Although the subjects considered here is merely a small sample of nonlinear Hartree equations, it is believed that the methods and results achieved for these Hartree equations can be applied or extended to more general cases. The nonlinear Hartree equations, including a large body of classes, are widely used models for a great number of problems in the fields of physics, chemistry and biology, and have gained a surge of attention from mathematicians ever since they were derived. In addition to mathematical analysis, the numeric of these equations is also a beautiful world and the studies on it have never stopped.

Keywords: nonlinear, hartree equations, problems, numerical methods, mathematicians

Introduction

Nonlinear dispersive and wave equations are fundamental models to many areas of physics and engineering like plasma physics, nonlinear optics, Bose-Einstein condensates, water waves, and general relativity. Examples include the nonlinear Schrondinger, wave, Klein-Gordon, water wave, and Einstein's equations of general relativity. This field of PDE has witnessed an explosion in activity in the past twenty, partly because of several successful cross-pollinations with other areas of mathematics; mainly harmonic analysis, dynamical systems, and probability. It also continues to be one of the most active areas of research, rich with problems and open to many interesting directions.

Although the numerical approximation of solutions of differential equations is a traditional topic in numerical analysis, has a long history of development and has never stopped, it remains as the beating heart in this field that to propose more sophisticated numerical methods for dispersive equations.

In this study, we consider the Cauchy problems concerning the relativistic Hartree equations:

$$\begin{cases} i\partial_t u = \sqrt{1-\Delta}u + F(u) \ \text{ in } \ \mathbb{R}^n \times \mathbb{R}, \ n \geq 3, \\ u(0) = \varphi, \end{cases} \tag{1}$$

$$\begin{cases} \partial_t^2 u + (1-\Delta)u = F(u) \ \text{ in } \ \mathbb{R}^n \times \mathbb{R}, \ n \geq 3, \\ u(0) = \varphi_1, \quad \partial_t u(0) = \varphi_2. \end{cases} \tag{2}$$

The nonlinear part F(u) is of Hartree type such that $F(u) = V_\gamma(u)u,$ where

$$V_\gamma(u)(x) = \lambda(|\cdot|^{-\gamma} * |u|^2)(x) = \lambda \int_{\mathbb{R}^n} \frac{|u(y)|^2}{|x-y|^\gamma}\, dy.$$

Here A is a non-zero real number and γ is a positive number less than the space dimension n.

The equation (1) is called the semi-relativistic equation which describes the Boson stars, and the second one (2) is the well-known Klein-Gordon equation whose physical model is originated from the helium atom - For the simplicity of presentation, the mass, speed of light and Planck constant of both equations have been normalized.
The equations **(1)** and (2) can be rewritten in the form of the integral equations

$$u(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')\,dt', \tag{3}$$

$$u(t) = (\cos t\omega)\varphi_1 + \omega^{-1}(\sin t\omega)\varphi_2 - \int_0^t \omega^{-1}(\sin(t-t')\omega)F(u)\,dt', \tag{4}$$

Where $\omega = \sqrt{1-\Delta}$ and the associated unitary group U(t) is realized by the Fourier transform as

$$U(t)\varphi = (e^{-it\omega}\varphi)(x) \equiv \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-it\sqrt{1+|\xi|^2}}\widehat{\varphi}(\xi)\,d\xi,$$

Where $\widehat{g}$ denotes the Fourier transform of g defined by $\widehat{g}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}g(x)\,dx.$
The operators cos $t\omega$ and sin $t\omega$ are defined by replacing $e^{-it\sqrt{1+|\xi|^2}}$ with $\cos(t\sqrt{1+|\xi|^2})$ and $\sin(t\sqrt{1+|\xi|^2})$, respectively. If the solution u of (1) or (3) has a decay at infinity and smoothness, it satisfies two conservation laws:

$$\begin{aligned} \|u(t)\|_{L^2} &= \|\varphi\|_{L^2}, \\ E_1(u) &\equiv K_1(u) + V(u) = E_1(\varphi), \\ K(u) = \frac{1}{2}\langle\sqrt{1-\Delta}\,u, u\rangle, &\; V(u) = \frac{1}{4}\langle F(u), u\rangle, \end{aligned} \tag{5}$$

Where $\langle , \rangle$ is the complex inner product in L^2. Also the solution of (2) or (4) or satisfies the conservation law:

$$\begin{aligned} E_2(u,\partial_t u) &= K_2(u,\partial_t u) - V(u) - E_2(\varphi_1,\varphi_2), \\ K_2(u,\partial_t u) &= \frac{1}{2}(\langle\partial_t u, \partial_t u\rangle - \langle\sqrt{1-\Delta}\,u, \sqrt{1-\Delta}\,u\rangle). \end{aligned} \tag{6}$$

The main concern of this study is to establish the global well-posedness and scattering of radial solutions of the equations (1) and
(2).
The study of the global well-posedness (GWP) and scattering for the semi- relativistic equation (1) has not been long before. In (E. Lenzmann) GWP was considered with a three dimensional Coulomb type potential which corresponds to $\gamma = 1$. The first and second authors of the present study showed GWP for $0 < \gamma \le 1$ if $n \ge 2$ and $0 < \gamma < 1$ if n=1, for $0 < \gamma < \frac{2n}{n+1}$ if $n \ge 2$,
and small data scattering for $\gamma > 2$ if $n \ge 3$ In this study we tried to fill the gap $1 < \gamma \le 2$ for GWP under the assumption of radial symmetry. For further study like blowup of solutions, solitary waves, mean field limit problem for semi-relativistic equation, see the references.

The first result is on the GWP for radial solutions of (3).

Theorem 2. Let $\gamma = 2$ and $n \geq 4$. Let $\varphi \in H^1$ be radially symmetric. If $\|\varphi\|_{H^1}$ is sufficiently small, then there exists a unique radial solution $u \in C_b H^1$ such that $|x|^{-1}u \in L^2L^2$ to Moreover, there exist radial functions φ^+ and φ^- such that

$$\int_{S^{n-1}} |re_1 - \rho\sigma|^{-\theta}\, d\sigma \leq M(r,\rho) < \infty,$$
$$\|u(t) - U(t)\varphi^{\pm}\|_{H^1} \to 0 \quad \text{as} \quad t \to \pm\infty$$

In (Y. Cho, *et al.* 2006), the authors used the L^qL^r type Strichartz estimates of the Klein-Gordon equation to prove GWP and scattering for the equation (1). Contrary to the case of Klein-Gordon equation, semi-relativistic equation preserves regularity in a

contraction argument based on the Strichartz estimate, from which the gap $(\frac{2n}{n+1} \leq \gamma \leq 2)$ arises naturally in the range of γ for GWP. To tide over this difficulty, we assume the radial symmetry for data and solutions, which enables us to estimate fractional

integrals associated with the nonlinearity $V_\gamma(u)u$. Then we establish an L^2 Strichartz estimate for $n \geq 2$ with weight $|x|^{-a}$ which is useful to treat radial functions but also applicable to non-radial functions (a gain of angular regularity is achieved in the non-radial case). The one dimensional analog is attainable.

For GWP we use a fractional integral estimate on the unit sphere such that

$$\int_{S^{n-1}} |re_1 - \rho\sigma|^{-\theta}\, d\sigma \leq M(r,\rho) < \infty,$$

where $e_1 = (1, 0, \cdots, 0)$. The result of Theorem E corresponds to the case $\theta = \gamma + \frac{1}{2}$

If n = 3, then the finiteness of integral enforces γ to be less than $\frac{3}{2}$ as in Theorem H since the integral is finite only when $n - 2 - \theta > -1$. For details see Lemma El and Lemma SI In Theorem El we treated the case $\theta = 2$ for which the integral

is not finite if n = 3. However, the three dimensional GWP can be slightly improved up to $\frac{5}{3}$ by using another Strichartz estimate on

a hybrid Sobolev space. It will be worthy of trying to fill the gap $\frac{5}{3} \leq \gamma \leq 2$ for n = 3.

The Klein-Gordon equation (4.2) was initially studied by (W. Strauss, 2001). In (T. Motai, 2008), the GWP is considered for $\lambda < 0$ and $0 < \gamma < 4$. It was proved that the scattering operator for (4.2) is well-defined on some domain if

$n \geq 2$, $4/3 < \gamma \leq 4n/(n+1)$ and $\gamma < n$. Furthermore, (K. Mochizuki, 2009) showed that if $n \geq 3$, $2 \leq \gamma \leq 4$ and $\gamma < n$, then the scattering operator can be defined on some neighborhood near zero in the energy space.

In this study the small data scattering of radial solutions is successfully treated below energy space, provided $\frac{3}{2} < \gamma < 2$. To state precisely, let us define a weighted spaces $W_{s,\varepsilon}$ and a data space $D_{\alpha,\beta}$ by

$$W_{s,\varepsilon} = \{\psi \in L^2 : \|\psi\|^2_{W_{s,\varepsilon}} \equiv \||\cdot|^{-s-\varepsilon}\psi\|^2_{L^2(|x|\leq 1)} + \||\cdot|^{-s+\varepsilon}\psi\|^2_{L^2(|x|>1)} < \infty\}$$
$$D_{\alpha,\beta} = H^{\alpha-\frac{1}{2}} \cap L^{\frac{2n}{n+2-2\beta}}$$

And , respectively, where $\varepsilon > 0$ is sufficiently small.

Theorem 3. Let $\frac{3}{2} < \gamma < 2$ for n = **3** and $\frac{3}{2} < \gamma \le 2$ for $n \ge 4$. Then there is a real number s and ε such that

$$\frac{1}{2} < s < \frac{\gamma}{2}, \quad 0 < \varepsilon < \min\left(\frac{\gamma}{2} - s, s - \frac{1}{2}\right), \quad 1 + s - \varepsilon < \gamma < 1 + s + \varepsilon. \tag{7}$$

where is the solutions to the Cauchy problem

$$\begin{cases} \partial_t^2 u^\pm + (1 - \Delta) u^\pm = 0, \\ u^\pm(0) = \varphi_1^\pm, \partial_t u^\pm(0) = \varphi_2^\pm. \end{cases} \tag{8}$$

In the definit ion of initial data space $D_{\alpha,\beta}$ the space $L^{\frac{2n}{n+2-2\beta}}$ can be slightly weak ened by the homogeneous Sobolev space $\dot{H}^{-(1-\beta)}$. In fact, $L^{\frac{2n}{n+2-2\beta}} \hookrightarrow \dot{H}^{-(1-\beta)}$ for $0 < \beta < 1$. Let $\widetilde{D}_{\alpha,\beta}$ be the weak ened space $H^{\alpha-\frac{1}{2}} \cap \dot{H}^{-(1-\beta)}$. Then one can easily show that the solution $u \in C_b(\mathbb{R}; \dot{H}^{-(1-(s-\varepsilon))})$ and then the existence of scattering operator of (2) on a small neighborhood of the origin in $\widetilde{D}_{s+\varepsilon,s-\varepsilon} \times \widetilde{D}_{s+\varepsilon-1,s-\varepsilon}$. For details see Remark M below.

The lower bound $\frac{3}{2}$ of γ is caused by the condition **(J?J)** which follows from the relation between the weight $|x|^{-a}$ and the L^2 estimate of Bessel function such that

$$\int_0^\infty r^{1-2a} |J_{\frac{n-2}{2}}(r)|^2 \, dr < \infty.$$

For the finiteness, the assumption $\frac{1}{2} < a < \frac{n}{2}$ is inevitable because $J_{\frac{n-2}{2}}(r) = O(r^{\frac{n-2}{2}})$ as $r \to 0$ and $J_{\frac{n-2}{2}}(r) = O(r^{-\frac{1}{2}})$ as $r \to \infty$. For more explicit formula, see the identity below. Hence for the present it seems hard to improve the range of γ for the small data scattering. From the perspective of negative result for the scattering[1], it will be very interesting to show the

scattering up to the value of γ greater than 1.

This study is organized as follows. In this Section we introduce a weighted Strichartz estimate for $n \ge 2$. Some fractional integral estimates are considered under radial symmetry.

If not specified, throughout this study, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \le CB$ and $A \ge C^{-1}B$, respectively. Different positive constants possibly de pending on n, λ, γ and a might be denoted by the same letter C. $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

Relativistic Hartree Equation for Boson Stars

The nonlinear relativistic Hartree equation was rigorously derived recently for a boson star, which refers to a quantum mechanical system of *N* bosons with relativistic dispersion interacting through a gravitational attractive or repulsive Coulomb potential In fact,

by starting from the TV-body relativistic Schrodinger equation (replacing —A/2 in the Schrodinger equation to

$\sqrt{-\Delta + m^2})$ and choosing the initial wave function to describe a condensate where *N* bosons are all in the same one-particle state, in the mean-field limit *N* oo, one can prove that the time evolution of the one-particle density is governed by the nonlinear relativistic Hartree equation (under a proper non-Dimensionalization). Also, one can refer to and references therein (with a slightly different dimensionless scaling in some cases) for overviews of other physical backgrounds.

It is easy to show that the equation admits at least two important conserved quantities, i.e. the *mass* of the system

$$N(\psi(\cdot,t)) := \|\psi(\cdot,t)\|^2 = \int_{\mathbb{R}^3} |\psi(\mathbf{x},t)|^2 \,\mathrm{d}\mathbf{x} \equiv \int_{\mathbb{R}^3} |\psi_0(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} = 1, \ t \geq 0, \quad (9)$$

And the *energy*

$$\begin{aligned} E(\psi(\cdot,t)) &:= \int_{\mathbb{R}^3} \left[\psi^* \left(-\Delta + m^2\right)^{1/2} \psi + \left(V_{\mathrm{ext}}(\mathbf{x}) + \frac{\lambda}{2|\mathbf{x}|} * |\psi|^2 \right) |\psi|^2 \right] \mathrm{d}\mathbf{x} \\ &\equiv E(\psi_0), \quad t \geq 0. \end{aligned} \quad (10)$$

The well-posedness of the initial-value problem was extensively and references therein. Their results can be summarized as: (i)

there exists a universal constant λ_{cr}(also referred to as the "Chandrasekhar limit mass" in physics and with a lower bound $\lambda_{\mathrm{cr}} > 4/\pi$) such that, when $\lambda > -\lambda_{\mathrm{cr}}$, the solution is globally well-posed in the energy space $H^{1/2}(\mathbb{R}^3)$ provided that $V \in L^3(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$; (ii) when $\lambda \leq -\lambda_{\mathrm{cr}}$, the solution is locally well-posed; and (iii) when $\lambda < -\lambda_{\mathrm{cr}}$, the solution will blow up in finite time, which indicates the "gravitational collapse" of boson stars when the effective "mass" exceeds the critical

value λ_{cr} . Another problem of interests is the existence and uniqueness of the ground state, which is defined as the minimizer of the following nonconvex minimization problem:

Find $\phi_g \in S = \left\{ \phi \mid \phi \in H^{1/2}(\mathbb{R}^3), \|\phi\|^2 = 1 \right\}$ such that

$$E_g := E(\phi_g) = \min_{\phi \in S} E(\phi). \quad (11)$$

If $V_{\mathrm{ext}}(\mathbf{x}) \equiv 0$, it was shown that the ground state exists iff $-\lambda_{\mathrm{cr}} < \lambda < 0$ and any ground state is smooth, decays exponentially when $|\mathbf{x}| \to \infty$, and is identical to its spherically symmetric rearrangement up to phase and translation. Moreover, it was proven recently that, when $\lambda < 0$ and $|\lambda| \ll 1$, the spherical-symmetric ground state is unique up to phase and translation, and the author remarked there that whether such uniqueness result can be extended to the whole range of existence $-\lambda_{\mathrm{cr}} < \lambda < 0$ remains open. Thus, such critical value λ_{cr} plays an important role in investigating the ground states and dynamics. One remark here is that based on numerical results $\lambda_{\mathrm{cr}} \approx 2.69 > 8/\pi$.

For the Schrodinger-Poisson (or -Newton) equations, i.e. the pseudo differential operator $\sqrt{-\Delta + m^2}$ in (1.8) is replaced by $-\Delta$, different numerical methods were presented in the literature based on finite difference discretization. However, these numerical methods have some difficulties in discretizing the 3D relativistic Hartree equation efficiently and accurately due to the appearance of the pseudo differential operator. The main aim of this

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chapter is to design efficient and accurate numerical methods for
computing the ground states and the dynamics of the initial-value problem. For this purpose, let $\beta = 4\pi\lambda$ and

$$V_P(\mathrm{x},t) = \frac{1}{4\pi|\mathrm{x}|} * |\psi|^2 = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathrm{x}-\mathrm{x}'|}|\psi(\mathrm{x}',t)|^2\,\mathrm{dx}', \quad \mathrm{x}\in\mathbb{R}^3, \quad t\geq 0,$$

Then equation is re-written as the relativistic Schrodinger-Poisson (RSP) equation

$$i\partial_t\psi(\mathrm{x},t) = \sqrt{-\Delta+m^2}\,\psi + V_{\mathrm{ext}}(\mathrm{x})\psi + \beta V_P\,\psi, \quad \mathrm{x}\in\mathbb{R}^3, \quad t>0, \qquad (12)$$

$$-\Delta V_P(\mathrm{x},t) = |\psi|^2, \quad \mathrm{x}\in\mathbb{R}^3, \quad \lim_{|\mathrm{x}|\to\infty} V_P(\mathrm{x},t) = 0, \quad t\geq 0. \qquad (13)$$

With this formulation, the energy functional (10) is re-written as

$$\begin{aligned} E(\psi(\cdot,t)) &= \int_{\mathbb{R}^3}\left[\psi^*\left(-\Delta+m^2\right)^{1/2}\psi + \left(V_{\mathrm{ext}}(\mathrm{x}) + \frac{\beta}{2}V_P\right)|\psi|^2\right]\mathrm{dx} \\ &= \int_{\mathbb{R}^3}\left[\left|\left(-\Delta+m^2\right)^{1/4}\psi\right|^2 + \left(V_{\mathrm{ext}}(\mathrm{x}) + \frac{\beta}{2}(-\Delta)^{-1}|\psi|^2\right)|\psi|^2\right]\mathrm{dx} \\ &= \int_{\mathbb{R}^3}\left[\left|\left(-\Delta+m^2\right)^{1/4}\psi\right|^2 + V_{\mathrm{ext}}(\mathrm{x})\,|\psi|^2 + \frac{\beta}{2}\,|\nabla V_P|^2\right]\mathrm{dx}, \quad t\geq 0. \end{aligned} \qquad (14)$$

In the spirit of observations drawn in this study, the problem is then truncated into a box with homogeneous Dirichlet boundary conditions and a backward Euler sine pseudo spectral method is applied to discretize it. For computing the dynamics, again the problem is truncated into a box with homogeneous Dirichlet boundary conditions and a time-splitting sine pseudo spectral method is applied to discretize it. In particular, when the potential and initial data for dynamics are spherically symmetric, the problem collapses to a quasi-ID problem.

Numerical Methods for Nonlinear Relativistic Hartree Equation

Efficient and accurate numerical methods are presented for computing ground states and dynamics of the three-dimensional (3D) nonlinear relativistic Hartree equation both without and with an external potential. This equation was derived recently for describing the mean field dynamics of boson stars. In its numeric, due to the appearance of pseudo differential operator which is defined in phase space via symbol, spectral method is more suitable for the discretization in space than other numerical methods such as finite difference method, etc. For computing ground states, a backward Euler sine pseudo spectral (BESP) method is proposed based on a gradient flow with discrete normalization; and respectively, for computing dynamics, a time-splitting sine pseudo spectral (TSSP) method is presented based on a splitting technique to decouple the nonlinearity. Both BESP and TSSP are efficient in computation via discrete sine transform, and are of spectral accuracy in spatial discretization. TSSP is of second-order accuracy in temporal discretization and conserves the normalization in discretized level. In addition, when the external potential and initial data for dynamics are spherically symmetric, the original 3D problem collapses to a quasi-1D problem, for which both BESP and TSSP methods are extended successfully with a proper change of variables. Finally, extensive numerical results are reported to demonstrate the spectral accuracy of the methods and to show very interesting and complicated phenomena in the mean field dynamics of boson stars.

In this study, we aim to design efficient and accurate numerical methods for computing ground states and dynamics of the three-dimensional (3D) nonlinear relativistic Hartree equation:

$$i\partial_t \psi = \sqrt{-\Delta + m^2}\,\psi + V(\mathbf{x})\psi + \lambda\left(|\mathbf{x}|^{-1} * |\psi|^2\right)\psi, \qquad \mathbf{x} \in \mathbb{R}^3, \quad t > 0, \tag{1}$$

With the following initial condition for dynamics

$$\psi(\mathbf{x}, 0) = \psi_0(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^3. \tag{2}$$

Here, t is time, x = (x, y, z)^T is the Cartesian coordinates, ψ = ψ(x, t) is a complex valued dimensionless wave function, a real-valued function V (x) stands for an external potential, m denotes the scaled particle mass (m = 1 in most cases) and λ ∈ R is a dimensionless constant describing the interaction strength. The sign of λ depends on the type of interaction: positive for the repulsive interaction and negative for the attractive interaction. The pseudo differential operator $\sqrt{-\Delta + m^2}$ for the kinetic energy is defined via multiplication in the Fourier space with the symbol $\sqrt{|\xi|^2 + m^2}$ for $\xi \in \mathbb{R}^3$, which is frequently used in relativistic quantum mechanical models as a convenient replacement of the full (matrix-valued) Dirac operator. The symbol ∗ stands for the convolution in R^3. In addition, the initial condition is usually normalized under the normalization condition by a proper non-Dimensionalization.

$$\|\psi_0\|^2 := \int_{\mathbb{R}^3} |\psi_0(\mathbf{x})|^2 \, d\mathbf{x} = 1. \tag{3}$$

The above nonlinear relativistic Hartree equation (1) was rigorously derived recently for a quantum mechanical system of N bosons with relativistic dispersion interacting through a gravitational attractive or repulsive Coulomb potential, which is often referred to as a boson star. It was achieved (under a proper non-Dimensionalization) in the mean field limit N → ∞ by choosing the initial wave function to describe a condensate where the N bosons are all in the same one-particle state, and is now used as a single-particle model for describing the mean field dynamics of boson stars. Also, we refer readers to references therein (with a slightly different dimensionless scaling in some cases) for other physical backgrounds of (1). It is easy to show that the equation (1.1) admits two important conserved quantities, i.e. the mass of the system

$$N(\psi(\cdot,t)) := \|\psi(\cdot,t)\|^2 = \int_{\mathbb{R}^3} |\psi(\mathbf{x},t)|^2 \, d\mathbf{x} \equiv \int_{\mathbb{R}^3} |\psi_0(\mathbf{x})|^2 \, d\mathbf{x} = 1, \qquad t \geq 0, \tag{4}$$

and the total energy,

$$\begin{aligned} E(\psi(\cdot,t)) &:= \int_{\mathbb{R}^3} \left[\psi^* \left(-\Delta + m^2\right)^{1/2} \psi + \left(V(\mathbf{x}) + \frac{\lambda}{2|\mathbf{x}|} * |\psi|^2\right) |\psi|^2\right] d\mathbf{x} \\ &\equiv E(\psi_0), \qquad t \geq 0, \end{aligned} \tag{5}$$

Where f^* denotes the complex conjugate of a function f. The well-posedness of the initial-value problem (1)-(2) was extensively studied and references therein. Their results are summarized as:

1. There exists a universal constant λcr (also referred to the "Chandrasekhar limit mass" in physics and with a lower bound λcr > 4/π) such that, when λ > −λcr, the solution is globally well-posed in the energy space H1/2 (R 3) provided that V ∈ L 3 (R 3) ∩L∞(R 3);
2. When λ ≤ −λcr, the solution is locally well-posed; and
3. when λ < −λcr, the solution will blow up in finite time, which indicates the "gravitational collapse" of boson stars when the effective 'mass' exceeds the critical value λcr. Another problem of interests is the existence and uniqueness of the ground state for (1), which is defined as the minimizer of the following nonconvex minimization problem:

Find $\phi_g \in S = \{\phi \mid \phi \in H^{1/2}(\mathbb{R}^3), \|\phi\|^2 = 1\}$ such that

$$E_g := E(\phi_g) = \min_{\phi \in S} E(\phi).$$

(6)

If V (x) ≡ 0, it was shown that the ground state exists iff −λcr < λ < 0 and any ground state is smooth, decays exponentially when |x| → ∞, and is identical to its spherically symmetric rearrangement up to phase and translation. Moreover, it was proven recently that, when λ < 0 and |λ| ≪ 1, the spherical-symmetric ground state is unique up to phase and translation, and the author remarked there that whether such uniqueness result can be extended to the whole range of existence −λcr < λ < 0 remains open. Thus, such critical value λcr plays an important role in investigating the ground states and dynamics of (1). We remark here that based on our numerical results λcr ≈ 2.69 > 8/π (cf. Fig).

For Schrodinger -Poisson (or -Newton) equations, i.e. the pseudo differential operator $\sqrt{-\Delta + m^2}$ in (1) is replaced by −∆, different numerical methods were presented in the literatures based on finite difference discretization; see, e.g.,. However, these numerical methods have some difficulties in discretizing the 3D relativistic Hartree equation efficiently and accurately due to the appearance of the pseudo differential operator. To our knowledge, there are almost no numerical results for the ground state and dynamics of the relativistic Hartree equation in the literatures. The main aim of this paper is to design efficient and accurate numerical methods for computing the ground state of (1.1) and the dynamics of the initial value problem (1)-(2). For this purpose,

let β = 4πλ and $\varphi(\mathbf{x},t) = \frac{1}{4\pi|\mathbf{x}|} * |\psi|^2 = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{x}'|}|\psi(\mathbf{x}',t)|^2\,d\mathbf{x}', \qquad \mathbf{x}\in\mathbb{R}^3, \quad t\geq 0,$

Then (1) is re-written as the relativistic Schrodinger -Poisson (RSP) system

$$i\partial_t\psi = \sqrt{-\Delta + m^2}\,\psi + V(\mathbf{x})\psi + \beta\varphi\psi, \qquad \mathbf{x}\in\mathbb{R}^3, \quad t>0, \tag{7}$$

$$-\Delta\varphi = |\psi|^2, \quad \mathbf{x}\in\mathbb{R}^3, \qquad \lim_{|\mathbf{x}|\to\infty}\varphi(\mathbf{x},t) = 0, \qquad t\geq 0. \tag{8}$$

With this formulation, the energy functional (5) is re-written as

$$\begin{aligned} E(\psi(\cdot,t)) &= \int_{\mathbb{R}^3}\left[\psi^*\left(-\Delta+m^2\right)^{1/2}\psi + \left(V(\mathbf{x})+\frac{\beta}{2}\varphi\right)|\psi|^2\right]d\mathbf{x} \\ &= \int_{\mathbb{R}^3}\left[\left|\left(-\Delta+m^2\right)^{1/4}\psi\right|^2 + \left(V(\mathbf{x})+\frac{\beta}{2}(-\Delta)^{-1}|\psi|^2\right)|\psi|^2\right]d\mathbf{x} \\ &= \int_{\mathbb{R}^3}\left[\left|\left(-\Delta+m^2\right)^{1/4}\psi\right|^2 + V(\mathbf{x})|\psi|^2 + \frac{\beta}{2}|\nabla\varphi|^2\right]d\mathbf{x} \equiv E(\psi_0), \quad t\geq 0. \end{aligned} \tag{9}$$

In order to design numerical method for computing the ground state, we first construct a gradient flow with discrete normalization (GFDN) which was widely and successfully used in computing ground states of Bose-Einstein condensation and the Schrodinger - Poisson-Slater equations. Then the problem is truncated into a box with homogeneous Dirichlet boundary conditions and a backward Euler sine pseudo spectral method is applied to discretize it. For computing the dynamics, again the problem is truncated into a box with homogeneous Dirichlet boundary conditions and a time-splitting sine pseudo spectral method is applied to discretize it. In particular, when the potential and initial data are spherically symmetric, then the problem collapses to a quasi-1D problem. Simplified numerical methods are designed by using a proper change of variables in the quasi-1D problem. The paper is organized as follows. A backward Euler sine pseudo spectral method is proposed for computing the ground state in 3D. A time-splitting sine pseudo spectral method is presented for computing the dynamics in 3D. In simplified numerical methods are

presented when the potential V (x) and initial data ψ0(x) are spherically symmetric. Extensive numerical results are reported in Section 5 to demonstrate the efficiency and accuracy of our numerical methods and to show the ground states and mean field dynamics of boson stars. Finally, some closing remarks are drawn. Throughout the study, we adopt the standard Sobolev spaces and their corresponding norms.

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Conclusion

Nonlinear dispersive and wave equations are fundamental models to many areas of physics and engineering like plasma physics, nonlinear optics, Bose-Einstein condensates, water waves, and general relativity. This field of PDE has witnessed an explosion in activity in the past twenty, partly because of several successful cross-pollinations with other areas of mathematics; mainly harmonic analysis, dynamical systems, and probability. It also continues to be one of the most active areas of research, rich with problems and open to many interesting directions. The goal of the present study is to continue our explorations of the effect of pe-riodicity on rough initial data for nonlinear evolution equations in the context of two important examples: the nonlinear Schrodinger (nlS) and Korteweg-deVries (KdV) equations, possessing, respectively, elementary second and third order monomial dispersion. The nonlinear dispersive equations, including a large body of classes, are widely used models for a great number of problems in the fields of physics, chemistry and biology, and have gained a surge of attention from mathematicians ever since they were derived. In addition to mathematical analysis, the numeric of these equations

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