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Study of the homology theory of fuzzy algebra

Alaa H. N. Mohamed & Mohsen A. H. Eid
Math. Dept., Faculty of science, Aswan University, Aswan. Egypt-
ala2222000@yahoo.com , scientist_samar@yahoo.com

Abstract:

In this article, the theorem of a universal coefficient of fuzzy homology modules is illustrated. By this result, we drawing the Mayer-Vietories sequence of fuzzy homology and allot several ensue on it.

Keywords: *Fuzzy algebra, homology theory.*

Mathematics Subject Classification: *17B55.*

Introduction

The ideas of a fuzzy algebra were presented in [7]. What's more, in [8], the fuzzy modules were introduced. And then, the category of fuzzy modules has examined in [10] and [6].

In [2] the scientists have presented (co)chain complexes of the fuzzy category. And they defined the fuzzy module and it`s an exact sequence. Furthermore, they delineated the outcomes utilizing in [5] and [6]. And in [1], Sadi B. and Cigdem G. construct the fuzzy homology module sequence under some conditions to prove the theorem of universal coefficients of fuzzy homology.

Here, we requisite to direct the underlying definitions and theorems which we utilized after. The references which testament use are [3], [4], [9], [10] and [11].

Definition 1:

A chain complex of a fuzzy module $\theta_C = \{\theta_{C_n}^n, \tilde{\partial}_n\}$ over Λ the object with the fuzzy endomorphism $\tilde{\partial} = \theta_C \to \theta_C$ with $\tilde{\partial}\tilde{\partial} = 0$ and $im\tilde{\partial}_{n+1} \subseteq ker\tilde{\partial}_n$.

Then we can define the (nth) fuzzy homology module as

$$H_n(\theta_C) = \bar{\theta}_n \; {}^{ker\partial_n}/_{im\partial_{n+1}}$$

Where $\bar{\theta}_n$ denotes the fuzzy quotient of $ker\tilde{\partial}_n$ by$im\tilde{\partial}_{n+1}$.

Theorem 1:

The additive functor $H_n(*)$ is the map

$$H_n(*): FComp \to * \, fzmod \qquad \forall\, n \in \mathbb{Z}. \text{ (See [2])}$$

Definition 2:

Consider the right fuzzy Λ-module μ_A and the left fuzzy Λ-moduleV_B. The fuzzy projective of μ_A is

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$$\bar{0} \to \mu_{0_R} \xrightarrow{\tilde{f}} \mu_{0_P} \xrightarrow{\tilde{g}} \mu_A \to \bar{0}$$

Then we define

$$F - Tor_{\tilde{g}}^{\Lambda}(\mu_A, V_B) = \ker\left(\tilde{f}_* = \tilde{f} \otimes \tilde{1}: (\mu_0 \otimes V)_{R\otimes_\Lambda} B \to (\mu_0 \otimes V)_{P\otimes_\Lambda} B\right)$$

Then the following fuzzy sequence is exist

$$\bar{0} \to F - Tor_{\tilde{g}}^{\Lambda}(\mu_A, V_B) \xrightarrow{\tilde{\iota}} (\mu_0 \otimes V)_{R\otimes_\Lambda} B \xrightarrow{\tilde{f}\otimes\tilde{1}} (\mu_0 \otimes V)_{P\otimes_\Lambda B} \xrightarrow{\tilde{g}\otimes\tilde{1}} (\mu \otimes V)_{A\otimes_\Lambda B} \to \bar{0}$$

Definition 3:

We can define the cohomology of a fuzzy module $H(\theta_C)$of fuzzy chain complex$\theta_C = \{\theta_{C_n}^n, \tilde{\partial}^n\}$, since the coboundary fuzzy operator $\tilde{\partial}: \theta_C \to \theta_C$ such that; $\tilde{\partial}\tilde{\partial} = 0$. Then

$$H(\theta_C) = \{H^n(\theta_C)\}; \quad H^n(\theta_C) = \bar{\theta}^n \left({ker\ \partial^n}/{im\partial^{n-1}} \right)$$

Example 1:

Let θ_A *and* v_B are right and left fuzzy in Λ-*fzmod*, respectively. And consider the fuzzy projective of θ_A as;

$$\bar{0} \to \theta_{0_R} \xrightarrow{\tilde{f}} \theta_{0_P} \xrightarrow{\tilde{g}} \theta_A \to \bar{0}$$

Then the fuzzy co-chain complex is,

$$\begin{array}{ccccccc} \bar{0} & \to & Hom_\Lambda(\theta_{0_P}, v_B) & \xrightarrow{\tilde{f}^*} & Hom(\theta_{0_R}, v_B) & \to & \bar{0} \\ & & \tilde{1}\| & & \tilde{1}\| & & \\ \bar{0} & \xrightarrow{\tilde{\partial}^{-1}} & C^0 & \xrightarrow{\tilde{\partial}^0} & C^1 & \xrightarrow{\tilde{\partial}^1} & \bar{0} \end{array}$$

Since, $\theta_{C_n}^n = \tilde{0} \quad \forall\, n \neq 0,1$ then we can compute the cohomology as

$$H^n(\theta_C) = \begin{cases} Hom_\Lambda(\theta_\Lambda, v_B) & if\ n = 0 \\ Ext_\Lambda(\mu_\Lambda, v_B) & if\ n = 1 \\ 0 & otherwise \end{cases}$$

Definition 4:

Let $\tilde{\varphi}$ and $\tilde{\psi}$ are two fuzzy chain maps since $\tilde{\varphi}, \tilde{\psi}: \theta_C \to V_D$, then the fuzzy homotopy $\tilde{\Sigma}: \tilde{\varphi} \to \tilde{\psi}$ is the morphism with the degree $+1$ of $\tilde{\Sigma}: \theta_C \to V_D$ $s.h.$ $\tilde{\psi} - \tilde{\varphi} = \tilde{\partial}\tilde{\Sigma} + \tilde{\Sigma}\tilde{\partial}$, then $\forall n \in \mathbb{Z}$; we have

$$\tilde{\psi}_n - \tilde{\varphi}_n = \tilde{\partial}_{n+1}\tilde{\Sigma}_n + \tilde{\Sigma}_{n-1}\tilde{\partial}_n$$

Proposition 1:

The map $H(\tilde{\varphi}) = H(\tilde{\psi}): H(\theta_C) \to H(V_D)$ is fuzzy map. Where $\tilde{\varphi}, \tilde{\psi}: \theta_C \to V_D$ are two fuzzy homotopy.

Theorem 2:

The sequence

$$\cdots \leftarrow H_{n-1}(\theta_{C^{`}}) \overset{\tilde{\partial}_*}{\leftarrow} H_n(\theta_{C^{``}}) \leftarrow H_n(\theta_C) \leftarrow H_n(\theta_{C^{`}}) \leftarrow \cdots \quad (1)$$

is the fuzzy homology exact sequence where the exact short sequence

$$\tilde{0} \rightarrow \theta_{C^{`}} \overset{\tilde{\imath}}{\rightarrow} \theta_C \overset{\tilde{P}}{\rightarrow} \theta_{C^{``}} \rightarrow \tilde{0} \quad (2)$$

is a fuzzy splitting sequence.

Proof:

Since the sequence (2) is the fuzzy splitting, then the fuzzy homomorphisms,

$$\tilde{\jmath}_n: \theta_{C_n}^n \rightarrow \theta_{C_n^{`}}^{`n}, \qquad \tilde{q}_n: \theta_{C_n^{``}}^{``n} \rightarrow \theta_{C_n}^n \qquad \forall n \in \mathbb{Z}$$

are existed and satisfy,

$$\tilde{\jmath}_n \circ \tilde{\imath}_n = 1_{\theta_{C_n^{`}}^{`n}}, \qquad \tilde{p}_n \circ \tilde{q}_n = 1_{\theta_{C_n^{``}}^{``n}}, \qquad \tilde{\imath}_n \circ \tilde{\jmath}_n + \tilde{q}_n \circ \tilde{p}_n = 1_{\theta_{C_n}^n}$$

Then,

$\forall n \in \mathbb{Z}; \tilde{d}_n = \tilde{\jmath}_{n-1} \circ \tilde{\partial}_n \circ \tilde{q}_n: \theta_{C_n^{``}}^{``n} \rightarrow \theta_{C_{n-1}^{`}}^{`n-1}$ and $\tilde{d} = \{\tilde{d}_n\}: \theta_{C^{``}}^{``n} \rightarrow \theta_{C^{`}}^{`}$ are the fuzzy homomorphisms of fuzzy modules and fuzzy chain complexes, respectively.

Let, $d = \{d_n: C_n^{``} \rightarrow C_{n-1}^{`}\}$, then

$$\begin{aligned} i_{n-2}(\partial_{n-1}^{`} d_n) &= (i_{n-1}\partial_{n-1}^{`}) j_{n-1}\partial_n q_n = \partial_{n-1}(i_{n-1}j_{n-1})\partial_n q_n = \partial_{n-1}(1_{C_{n-1}} - q_{n-1}p_{n-1})\partial_n q_n \\ &= \partial_{n-1}\partial_n q_n - \partial_{n-1}q_{n-1}p_{n-1}\partial_n q_n = -\partial_{n-1}q_{n-1}p_{n-1}\partial_n q_n \\ &= -\partial_{n-1}q_{n-1}(p_{n-1}\partial_n)q_n = -\partial_{n-1}q_{n-1}\partial_n^{``}p_n q_n = -\partial_{n-1}q_{n-1}\partial_n^{``}1_{C_n^{``}} = -\partial_{n-1}q_{n-1}\partial_n^{``} \\ &= -(i_{n-2}j_{n-2} + q_{n-2}p_{n-2})\partial_{n-1}q_{n-1}\partial_n^{``} \\ &= -i_{n-2}(j_{n-2}\partial_{n-1}q_{n-1})\partial_n^{``} - q_{n-2}(p_{n-2}\partial_{n-1})q_{n-1}\partial_n^{``} \\ &= -i_{n-2}(d_{n-1}\partial_n^{``}) - q_{n-2}\partial_{n-1}^{``}(p_{n-1}q_{n-1})\partial_n^{``} = -i_{n-2}(d_{n-1}\partial_n^{``}) \end{aligned}$$

Then we get $\partial_n^{`} d_n = d_{n-1}\partial_n^{``}$ since i_{n-2} is a monomorphism. And

$$\forall \ [\mathbb{Z}] \in H_n(C^{``}), \partial_{*_n}(\mathbb{Z}) = [i_{n-1}^{-1} \circ \partial_n \circ j_n^{-1}(\mathbb{Z})] = [j_{n-1} \circ \partial_n \circ q_n(\mathbb{Z})] \ = [d_n(\mathbb{Z})] = d_{n^*}[\mathbb{Z}]$$

Then the map $\tilde{\partial}_{*_n}: H_n(\theta_{C^{``}}) \rightarrow H_{n-1}(\theta_{C^{`}})$ is a fuzzy homomorphism. Then (2) is an exact sequence.

Proposition 2:

The short exact sequence

$$\tilde{0} \rightarrow \theta_{A^{`}} \overset{\tilde{\alpha}}{\rightarrow} V_A \rightarrow \eta_{A^{``}} \rightarrow \tilde{0}$$

is split fuzzy sequence with $\mu_{B^{`}}$, then the short exact sequence

$$\tilde{0} \rightarrow \theta_{A`} \otimes \mu_B \xrightarrow{\tilde{\alpha}\otimes\tilde{1}} V_A \rightarrow \mu_B \rightarrow \eta_{A``} \otimes \mu_B \rightarrow \tilde{0}$$

is split fuzzy sequence.

Proof:

From the definition of the tensor product. Since $\theta_C = \{\theta_{C_n}^n, \tilde{\partial}_n\}$ $\quad and\ for\ \mu_G$ we have, $\theta_C \otimes \mu_G = \{\theta_{C_n}^n \otimes \mu_G, \tilde{\partial}_n \otimes \tilde{1}_{\mu_G}\}$.

Definition 5:

Consider θ_C be a fuzzy chain complex, then the fuzzy homology is the homology with coefficients μ_G and denoted by $H_n(\theta_C, \mu_G)$.

Theorem 3:

From the split fuzzy sequence of the short exact sequence with μ_G

$$\tilde{0} \rightarrow \theta_{C`}^{`} \rightarrow \theta_C \rightarrow \theta_{C``}^{``} \rightarrow \tilde{0}$$

We get the fuzzy homology exact sequence

$$\cdots \leftarrow H_{n-1}(\theta_{C`}^{`}; \mu_G) \leftarrow H_n(\theta_{C``}^{``}; \mu_G) \leftarrow H_n(\theta_C; \mu_G) \leftarrow H_n(\theta_{C`}^{`}; \mu_G) \leftarrow \cdots$$

Theorem 4:

Let the free fuzzy chain complex θ_C and fuzzy module μ_G, then the sequence

$$0 \rightarrow H_n(\theta_C) \otimes \mu_G \xrightarrow{\tilde{\varphi}_n} H_n(\theta_C \otimes \mu_G) \rightarrow F - Tor(H_{n-1}(\theta_C), \mu_G) \rightarrow 0$$

is split.

Proof:

Consider two sub-complexes $\theta_{Z(C)} = \{ker\tilde{\partial}_n \subset \theta_{C_n}\}$ and

$\theta_{B(C)} = \{ker\tilde{\partial}_n \subset \theta_{C_{n-1}}\}$ of θ_C are free fuzzy chain complexes with the fuzzy homomorphisms $\tilde{\alpha}_n: \theta_{Z_n(C)} \rightarrow \theta_{C_n}, \quad \tilde{\beta}_n: \theta_{C_n} \rightarrow \theta_{B_{n-1}(C)}$. The short exact sequence

$$\tilde{0} \rightarrow \theta_{Z(C)} \xrightarrow{\tilde{\alpha}} \theta_C \xrightarrow{\tilde{\beta}} \theta_{B(C)} \rightarrow \tilde{0}$$

is exact. And we define the fuzzy homomorphism $\tilde{h}_n: \theta_{B_{n-1}(C)} \rightarrow C_{C_n}$ such that; $\tilde{\partial}_n \circ \check{h}_n = \tilde{1}_{\theta_{n-1}(C)}$. Then the map $h_n \otimes \tilde{1}_{\mu_G}: \theta_{B_{n-1}(C)} \otimes \mu_G \rightarrow \theta_{C_n} \otimes \mu_G$ define

$$F - Tor(\theta_C; \mu_G) \rightarrow H_n(\theta_C; \mu_G)$$

That is the inverse fuzzy homomorphism of

$$H_n(\theta_C; \mu_G) \rightarrow F - Tor(\theta_C; \mu_G)$$

Definition 6:

Let $\theta_C, \bar{\theta}_{\bar{C}}$ and $\tilde{\tau}$ are the notations for the fuzzy chain complex, free fuzzy complex and fuzzy homomorphism, respectively. Then $\tilde{\tau}: \bar{\theta}_{\bar{C}} \to \theta_C$ where, $\tilde{\tau}_*: H(\bar{\theta}_{\bar{C}}) \to H(\theta_C)$.

Theorem 5:

Let θ_C be a fuzzy complex, θ_A and θ_B are sub complexes of θ_C such that, $\theta_C = \theta_A \cup \theta_B$. If we define the fuzzy homomorphisms

$\tilde{\imath}_*: H_*(\theta_A \cap \theta_B) \to H_*(\theta_A) \oplus H_*(\theta_B)$ and $\tilde{\jmath}_*: H_*(\theta_A) \oplus H_*(\theta_B) \to H_*(\theta_C)$ as $\tilde{\imath}_*(\gamma) = (\tilde{\imath}_{1*}(\gamma), -\tilde{\imath}_{2*}(\gamma))$, $\tilde{\jmath}_*(\gamma_1, \gamma_2) = \tilde{\jmath}_{1*}(\gamma_1) + \tilde{\jmath}_{2*}(\gamma_2)$ where

$\tilde{\imath}_1: \theta_A \cap \theta_B \to \theta_A, \tilde{\imath}_2: \theta_A \cap \theta_B \to \theta_B$, $\tilde{\jmath}_1: \theta_A \to \theta_C$, $\tilde{\jmath}_2: \theta_B \to \theta_C$ such that $\tilde{\jmath}_* \tilde{\imath}_* = 0$. Then we note that $ker\tilde{\jmath}_* = im\tilde{\imath}_*$ and by define fuzzy homomorphisms $\tilde{\partial}_*: H_*(\theta_C) \to H_*(\theta_A \cap \theta_B)$ and $\tilde{h}_*: H_*(\theta_B, \theta_A \cap \theta_B) \to H_*(\theta_C, \theta_A)$. Then we get the fuzzy long exact sequence

$$\begin{array}{ccccccc} H_n(\theta_A \cap \theta_B) & \xrightarrow{\tilde{\imath}_{2*}} & H_n(\theta_B) & \xrightarrow{\tilde{k}_*} & H_*(\theta_B, \theta_A \cap \theta_B) & \xrightarrow{\tilde{\partial}} & H_{n-1}(\theta_A \cap \theta_B) \\ \tilde{\imath}_{1*} \downarrow & & \tilde{\jmath}_{2*} \downarrow & & \tilde{h}_* \downarrow & & \tilde{\imath}_{1*} \downarrow \\ H_n(\theta_A) & \xrightarrow{\tilde{\jmath}_{1*}} & H_n(\theta_C) & \xrightarrow{\tilde{l}_*} & H_n(\theta_C, \theta_A) & \to & H_{n-1}(\theta_A) \end{array}$$

Since $\tilde{\partial}_* \circ (\tilde{h}_*)^{-1} \circ \tilde{l}_*: H_n(\theta_C) \to H_{n-1}(\theta_A \cap \theta_B)$.

Theorem6:

Consider θ_C, M and $\mathcal{M}_r(M)$ are a fuzzy module bimodule over θ_C and the matrices of $r \times r$ degree, respectively. Then, $\forall\, r \geq 1$, we have

$$tr_*: H_*(\mathcal{M}_r(\theta_C), \mathcal{M}_r(M)) \to H_*(\theta_C, M)$$

and

$$inc_*: H_*(\theta_C, M) \to H_*(\mathcal{M}_r(\theta_C), \mathcal{M}_r(M))$$

which are inverse to each other.

Proof:

Now, we need to prove that $(inc \circ tr)$ and (id) are homotopic. Consider the presimplicial homotopy $\tilde{h}$ where;

$$\tilde{h}: \sum (-1)^i \tilde{h}_i, \quad \tilde{h}_i: \mathcal{M}_r(M) \otimes \mathcal{M}_r(\theta_C)^{\otimes n} \to \mathcal{M}_r(M) \otimes \mathcal{M}_r(\theta_C)^{\otimes n+1}$$

Since

$$\tilde{h}_i(a^0, \dots, a^n) = \sum E_{j1}(a^0_{jk}) \otimes E_{11}(a^1_{km}) \dots \otimes E_{11}(a^i_{pq}) \otimes E_{1q}(1) \otimes a^{i+1} \otimes \dots \otimes a^n$$

Such that, $a^0 \in \mathcal{M}_r(M), otherwise\ a^s \in \mathcal{M}_r(\theta_C)$, and $\tilde{h}_i$ satisfy that

(1) $\tilde{d}_i \tilde{h}_i = \tilde{h}_{i-1} \tilde{d}_i \qquad if\ \ i < j,$

(2) $\tilde{d}_i\tilde{h}_i = \tilde{d}_i\tilde{h}_{i-1} \quad if\ 0 < i < n, i = j, j+1,$

(3) $\tilde{d}_i\tilde{h}_j = \tilde{h}_j\tilde{d}_{i-1} \quad if\ i > j+1.$

Where $\tilde{h} = \sum_{i=0}^{n}(-1)^i\tilde{h}_i$ and for $n = 0$, $\tilde{h}(a) = E_{j1}(a_{jk}) \otimes E_{1k}(1)$. If $n = 1$, $\tilde{h}(a,b) = E_{j1}(a_{jk}) \otimes E_{1k}(1) \otimes b - E_{j1}(a_{jk}) \otimes E_{11}(b_{ki})E_{1l}(1)$.

$$\therefore \tilde{h}\tilde{d} + \tilde{d}\tilde{h} = \tilde{d}_0\tilde{h}_0 - \tilde{d}_{n+1}\tilde{h}_n \ \text{ s.h. } \ id = \tilde{d}_0\tilde{h}_0, \quad \tilde{d}_{n+1}\tilde{h}_n = inc \circ tr$$

Then we get the required.

Reference:

[1] S. Bayramov, & C. Gunduz, "*The Universal Coefficient Theorem For Fuzzy Homology Modules*", ***International Science Press***, (2011), 41-50.

[2] N. Bourbaki, , *"Algebre Homologique*", ***Masson-Paris New york Barcelone*** Milan, 1980.

[3] H. N. Alaa, & Gouda, Y. Gh, *"On the dihedral (Co)homology for schemes",* ***International Electronic Journal of Algebra***, volume **5** (2009) 106-113.

[4] S. R. Lopez-Permouth, *"Lifting Morita Equivalence to Categories of Fuzzy Modules"*, **Information Sciences**, 64, (1992), 191-201.

[5] S.R. Lopez-Permouth, & D. S. Malik, *"On Categories of Fuzzy Modules",* **Information Sciences**, 72, (1993), 65-82.

[6] C. V. Negoita, and Ralescu, D.A., *"Application of Fuzzy Subsets to System Analysis",* **Birkhauser *Basel***, 1975.

[7] A. Rosenfeld, "Fuzzy Groups", ***J. Math. Anal. Appl***.,35,(1971),512-517.

[8] R. Jose, "*Mayer-Vietoris sequence in Cohomology of Lie Algebroids on Simplicial complexes*", Commun. ***Korean Math. Soc***., 2018, No.1357-1366.

[9] M.M. Zahedi and R.A. Ameri *"Fuzzy Flat and Faithfully Flat R-modules",* ***J. Fuzzy Math***., **5 (4)**, (1997), 855-864.

[10] M.M. Zahedi and R.A. Ameri *"Fuzzy Exact Sequence in Category of Fuzzy Modules"*, ***J. Fuzzy Math*., 2 (2)**, (1994), 409-424.

[11]M.M. Zahedi and R.A. Ameri *"On Fuzzy Projective and Injective Modules"*, ***J. Fuzzy Math***., **3 (1)** (1995), 181-190.