International Journal of Research
Available at https://journals.pen2print.org/index.php/ijr/
e-ISSN: 2348-6848
p-ISSN: 2348-795X
Volume 06 Issue 10
September 2019

Solution of Population Growth and Decay Problems by Using Laplace Transform

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Abstract:

Laplace transform are invaluable for any engineer's mathematical toolbox as they make solving linear Ordinary Differential Equations and related initial value problems as well as systems of linear Ordinary Differential Equations, much easier. The population growth and decay problems generally appear in the field of chemistry, biology, physics, social science etc. In this present paper, we used laplace transform for solving population growth and decay problems and in application section, some numerical applications are given to explain the usefulness of laplace transform for solving population growth and decay problems.

Keywords

Laplace transform, Inverse Laplace transform, Population growth and decay problems, Half- life.

1. Introduction

Mathematically, the population growth (growth of a city, growth of a plant or bacteria or a cell or a species or an organ) by the first order linear ordinary has differential equation.

$$\frac{dM}{dt} = kM - - - - - - (1)$$

With initial condition as

$$M(t_0) = M_0 - - - - - - - - (2)$$

Where k is a positive real number, M and M_0 are the amount of populations at time t and initial timet_0.

Equation (1) is known as the Malthusian law of population growth.

On the other hand the decay problem of the material (radioactive substance) is mathematically defined by the first order linear ordinary differential equation.

$$\frac{dM}{dt} = -kM - - - - - - (3)$$

with initial condition as

$$M(t_0) = M_0 - - - - - - - - (4)$$

Where M is the amount of the material at time t, k is a positive real number and M_0 is the amount of the material at initial timet_0.

In equation (3), the negative sign in the R.H.S is taken because the mass of the substance is decreasing with time and so the derivative $\frac{dM}{dt}$ must be negative. The laplace transform of the function F (t) for all $t \geq 0$ is defined as:

$$L[F(t)] = \int_0^{\alpha} F(t) e^{-st} \, dt = f(s)$$

Where L is Laplce transform operator.

Here we must assume that f(t) is such that the integral exists (that is, has some finite valve).The laplace transform of the function F (t) for $t \geq 0$ exist if F(t) is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of laplace transform of the function F (t).

2. Linearity Property of Laplace Transform

If

$L[F(t)] = f(s)$ and $L[G(t)] = g(s)$

then

$$L[a F(t) + b G(t)] = aL[F(t)] + bL[G(t)]$$
$$\Rightarrow L[aF(t) + b G(t)] = af(s) + b g(s)$$

where a, b are arbitrary constants.

2.1. Laplace Transform of Some Elementary Functions

Table 1. Laplace Transform.

No	F(t)	$L[F(t)] = f(s)$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^3}$
4	t^n (n=0,1,…..)	$\frac{n!}{s^{n+1}}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\frac{1}{s-a}$
7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
9	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
10	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$

2.2. Inverse Laplace Transform

If $L[F(t)] = f(s)$ then F (t) is called the inverse laplace transform of f(s) and mathematically it is defined as
$F(t) = L^{-1}[f(s)]$
Where L^{-1} is the inverse laplace transform operator.

Table 2. Inverse Laplace Transform.

No	f(s)	$F(t) = L^{-1}[f(s)]$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^2}$	t
3	$\frac{1}{s^3}$	$\frac{t^2}{2!}$
4	$\frac{1}{s^{n+1}}, (n = 0,1,2 \ldots.)$	$\frac{t^n}{n!}$
5	$\frac{1}{s^{n+1}}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6	$\frac{1}{s-a}$	e^{at}
7	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$
8	$\frac{1}{s^2+\omega^2}$	$\frac{\sin\omega t}{\omega}$
9	$\frac{s}{s^2-\omega^2}$	$\cosh \omega t$
10	$\frac{1}{s^2-\omega^2}$	$\frac{\sinh\omega t}{\omega}$

2.3. Laplace Transform of Derivate of the Function F(t)

If $L[F(t)] = f(s)$ then

1. $L[F'(t)] = sf(s) - F(0)$
2. $L[F''(t)] = s^2 f(s) - sF(0) - F'(s)$
3. $L[F^n(t)] = s^n f(s) - s^{n-1}F(0) \ldots - F^{(n-1)}(0)$

3. Laplace Transform for Population Growth Problem

We present laplace transform for population growth problem given by (1) and (2).

Applying the laplace transform on both sides of (1), we have

$$L\left[\frac{dM}{dt}\right] = kL\,[M(t)] \text{ --------(5)}$$

Now applying the property, laplace transform of derivative of function, on (5),

We have

$$sL[M(t)] - M(0) = kL\,[M(t)] \dots\dots (6)$$

Using (2) and (6) and on simplification

$$(s-k)L[M(t)] = M_0$$

$$\Rightarrow \; L\,[M(t)] = \frac{M_0}{(s-k)} = ------(7)$$

Operating inverse Laplace transform on both side of (7), we have

$$M(t) = L^{-1}\left[\frac{M_0}{(s-k)}\right]$$

$$\Rightarrow M(t) = M_0\, L^{-1}\left[\frac{1}{(s-k)}\right]$$

$$\Rightarrow M(t) = M_0\, e^{kt} ----- (8)$$

Which is the required amount of the population at time t.

4. Laplace Transform for Population Decay Problem

We present laplace transform for decay problem which is mathematically given by (3) and (4).

Applying the laplace transform on both sides of (3), we have

$$L\left[\frac{dM}{dt}\right] = -kL[M(t)] ------ (9)$$

Laplace transform of derivative of function on (9), we have

$$sL[M(t)] - M(0) = -kL\,[M(t)] -- (10)$$

Using (4) in (10) and on simplification

$$(s+k)\,L\,[M(t)] = 0$$

$$\Rightarrow \; L\,[M(t)] = \frac{M_0}{(s+k)} = -----(11)$$

Operating inverse laplace transform on both sides of (11), we have

$$M(t) = L^{-1}\left[\frac{M_0}{(s+k)}\right]$$

$$\Rightarrow M(t) = M_0\, L^{-1}\left[\frac{1}{(s+k)}\right]$$

$$\Rightarrow M(t) = M_0\, e^{-kt} ----- (12)$$

Which is the required amount of substance at time t.

5. Applications

In this section, some applications are given to explain the usefulness of laplace transform for solving population growth and decay problems. The aim of this work is to finding the solution of population growth and decay problems using laplace transform without large computational work.

Application(1): The population of Singapore grows at a rate proportional to the number of people presently living in Singapore. At after three years, the population has doubled, and after six years the population is 20,000, then estimate the number of people initially living in Singapore.

Mathematically the above problem can be written as

$$\frac{dM(t)}{dt} = k\,M(t) ----(13)$$

Where M denote the number of people living in the country at any time t and k is the constant of proportionality. Consider M_0 is the number of people initially living in Singapore at t = 0.

Taking the laplace transform on both sides of (13), we have

$$L\left[\frac{dM}{dt}\right] = kL[M(t)] ------(14)$$

Now applying the property, laplace transform of derivative of function on (14),

We have

$$sL[M(t)] - M(0) = kL\,[M(t)] --(15)$$

Since at t = 0, M = M_0, using in (15),

We have

$$(s-k)\,L\,[M(t)] = M_0$$

$$\Rightarrow \; L\,[M(t)] = \frac{M_0}{(s-k)} = -----(16)$$

Operating inverse laplace transform on both sides of (16), we have

$$M(t) = L^{-1}\left[\frac{M_0}{(s-k)}\right]$$

$$\Rightarrow M(t) = M_0\, L^{-1}\left[\frac{1}{(s-k)}\right]$$

$$\Rightarrow M(t) = M_0\, e^{kt} ----- (17)$$

Now at t = 3, M = $2M_0$, using in (17),

We have

$$2\,M_0 = M_0\, e^{3k}$$

$\Rightarrow\ e^{3k} = 2$

$\Rightarrow\ k=\frac{1}{3}\ln 2 = 0.231-----(18)$

Now using the condition at t=6, N=20,000 in (17), We have

$20{,}000 = M_0 e^{6k} ---------(19)$

Putting the value of k from (18) in (19), we have
$20{,}000 = M_0 e^{6x\ 0.231}$

$\Rightarrow 20{,}000 = 3.999 M_0$

$\Rightarrow M_0 \cong 5001 ---------(20)$

Which are the required number of people initially living in the Singapore.

Application(2): A radioactive material is known to decay at a rate proportional to the amount present. If initially there is 500 milligrams of the radioactive material present and after three hours it is observed that the radioactive material has lost 20 percent of its original mass, find the half-life of the radioactive material,

In Mathematical form, we can be written:

$\frac{dM(t)}{dt}$= - k M(t) $- -- ---(21)$

Where M denote the amount of radioactive material at time t and k is the constant of proportionality. Consider M_0 is the initial amount of the radioactive material at time $t = 0.$

Applying the laplace transform on both sides of (21), we have

$L\left[\frac{dM(t)}{dt}\right]$ = -k $L[M(t)] -- --(22)$

Now applying the property, laplace transform of derivative of function on (22),

We have

$sL[M(t)] - M(0) = -k L[M(t)] ----(23)$

Since at t=0, $M=M_0 = 500,$ so using in (23),
We have

$s\,L[M(t)] - 500 = -k\,L[M(t)]$

$\Rightarrow (s + k)\,L[M(t)] = 500$

$\Rightarrow\ L[M(t)] = \frac{500}{(s+k)} ------ (24)$

Operating inverse laplace transform on both sides of (24), we have

$M(t) = L^{-1}\left[\frac{500}{(s+k)}\right]$

$\Rightarrow M(t) = 500\ L^{-1}\left[\frac{1}{(s+k)}\right]$

$\Rightarrow M(t) = 500\ e^{-kt} ------(25)$

Now at t=3, the radioactive material has lost 20 percent of its original mass 500 mg so M=500-120 = 380, using in (25)

We have

$380 = 500 e^{-3k}$

$\Rightarrow e^{-3k} = 0.76$

$\Rightarrow k = -\frac{1}{3}\text{Ln } 0.76 \cong 0.09147 ---- (26)$

We required t when $M = \frac{M_0}{2} = \frac{500}{2} = 250$ so from (25), we have

$250 = 500\ e^{-kt} ----(27)$

Putting the value of K from (26) in (27)

we have

$250 = 500\ e^{-0.09147t} ----(28)$

$\Rightarrow e^{-0.09147t} = 0.5$

$\Rightarrow\ t= -\frac{1}{0.09149}\text{Ln } 0.5$

$\Rightarrow\ t \cong 7.577$ hours $---- (28)$

Which is required half- time of the radioactive material.

6. Conclusion

In the present paper, we have successfully discussed the laplace Transform for solving the population growth and decay problems. The given in application section show that the usefulness of laplace transform for solving population growth and decay problems. In the future, we can easily solve the problems of mechanics and electrical circuit using the present scheme.

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