



A Class of Generalized Closed Set in General Topology

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Abstract:

The purpose of this paper is to introduce the concept of a class of generalized closed set in an ideal topological space and to show some of its basic properties. In this paper we proved some equivalent conditions.

Key words:

Ideal; generalized closed set; local functions; expansion operator

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1. Introduction

At present many researchers are investigating different types of generalized closed sets and their various properties in general topology which leads use of general topology and bitopology in different branches of science. Fuzzy version of different types of generalized closed sets are being investigated in fuzzy topology.

M.S. El-Naschie [7,8] showed importance and uses of general topology and generalized closed sets in quantum physics, high energy physics and superstring theory of physics.

N. Levine [2] introduced the concept of generalized closed set (briefly *g-closed set*). An ideal I in a topological space (X, τ) is a non-empty collection of subsets of X satisfying the following two properties:

- (1) $A \in I$ and $B \subset A$ implies $B \in I$.
- (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I is called ideal topological space and it is denoted by (X, τ, I) . Kuratowski [3] defined

local function of A with respect to I and τ (briefly A^*) as for a subset $A \subseteq X$, $A^*(I) = \{x \in X \mid U \cap A \notin I \text{ for every neighbourhood } U \text{ of } x\}$.

Kasahara [5] defined γ operation on the topology τ on a given topological space (X, τ) is a function $\gamma: \tau \rightarrow P(X)$ such that $V \subseteq \gamma V$ for each $V \in \tau$. Tong [13] defined the γ -operation as *expansion*. The following operators are examples of the operation γ viz. the closure operator γ_{cl} defined by $\gamma(U) = Cl(U)$, the identity operator γ_{id} defined by $\gamma(U) = U$ and the interior-closure operator γ_{ic} defined by $\gamma(U) = Int(Cl(U))$.

Definition 1.1. (Levine [2]) A subset A of a topological space (X, τ) is called generalized closed set (briefly *g-closed*) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Definition 1.2. (Dontchev et.al. [1]) (I, γ) -generalised closed [briefly (I, γ) -*g-closed*] if $A^* \subseteq U^{\gamma}$ whenever $A \subseteq U$ and $U \in \tau$.

I-g-closed if $A^* \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$.

Dontchev et.al. [1] discussed one lemma. The lemma is -

Lemma 1.1. (Dontchev et.al. [1], lemma 2.6) If A and B are the subsets of (X, τ, I) then $(A \cap B)^* \subseteq A^* \cap B^*$.

2. (γ, δ) -*g-closed set*

Definition 2.1. A subset A of an ideal topological space (X, τ, I) is said to be (γ, δ) -*g-closed set* (briefly (γ, δ) -*gc*) if $U^{\gamma} \subseteq A^* \subseteq U^{\delta}$ whenever $A \subseteq U$ and $U \in \tau$. A is said to be an (γ, δ) -*go* if $(X \setminus A)$ is an (γ, δ) -*gc*. We call $(X, \tau, I, \gamma, \delta)$ as dual operation ideal topological space (DOITS).



Let $(X, \tau, I_1, \gamma, \delta)$ and $(Y, \vartheta, I_2, \zeta, \eta)$ be two DOITS. Then $(A \times B) \subseteq X \times Y$ is said to be $(\gamma_\delta, \zeta_\eta)$ -gc if $U^\gamma \times V^\zeta \subseteq (A \times B)^* \subseteq U^\delta \times V^\eta$ whenever $(A \times B) \subseteq U \times V$; $U \times V \in \mu \times \rho$; where $\mu \times \rho$ is product topology of $X \times Y$.

We define ideal in product topological space as if I_1 and I_2 are ideals of ideal topological spaces (X, τ, I_1) and (Y, ϑ, I_2) respectively then $I_1 \times I_2 = \{A \times B \mid A \in I_1, B \in I_2\}$ is ideal of product topological space $(A \times B, \mu \times \rho)$; where $\mu \times \rho$ is the product topology.

Hence we define $(A \times B)^* = \{(x, y) \mid (U \times V) \cap (A \times B) \notin M \times N, M \in I_1, N \in I_2; x \in U, y \in V; U \in \tau, V \in \vartheta\}$.

The set of all (γ, δ) -gc of DOITS $(X, \tau, I, \gamma, \delta)$ is denoted by $gc(X)$.

Example 2.1. Let A be a subset of a DOITS $(X, \tau, I, \gamma, \delta)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $I = \{\emptyset, \{c\}\}$; let us define γ, δ operator such that $U^\gamma = U$ and $U^\delta = Cl(U)$

Consider $A = \{a, b\}$, and $U = \{a, b\}$, then clearly $A \subseteq U$ and $U \in \tau$. Then we have $U^\gamma = \{a, b\}$ and $U^\delta = Cl(U) = X$. Using the definition of $A^*(I)$, $A^* = \{a, b, c\}$.

Next we consider $U = \{a, b, c\}$, then clearly $A \subseteq U$ and $U \in \tau$. Then we have $U^\gamma = X$ and $U^\delta = Cl(U) = X$. So, $U^\gamma \subseteq A^* \subseteq U^\delta$ whenever $A \subseteq U$ and $U \in \tau$. Thus $A = \{a, b\}$ is a $(\gamma_{id}, \delta_{cl})$ -gc

Theorem 2.1. Let $(X, \tau, I, \gamma, \delta)$ be DOITS. If A and B are (γ, δ) -gc then $(A \cup B)$ is also (γ, δ) -gc.

Proof. Let $(A \cup B) \subseteq U$ and $U \in \tau$. Then $A \subseteq U$ and $B \subseteq U$. As A and B are (γ, δ) -gc, so $U^\gamma \subseteq A^* \subseteq U^\delta$ and $U^\gamma \subseteq B^* \subseteq U^\delta$. Now we know that $(A \cup B)^* = A^* \cup B^*$. Hence $U^\gamma \subseteq (A \cup B)^* \subseteq U^\delta$. Hence result.

Theorem 2.2. Subset of a (γ, δ) -gc need not to be a (γ, δ) -gc.

For the above theorem; we have the following example.

Example 2.2. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $I = \{\emptyset, \{c\}\}$. Let us define γ, δ operator such that $U^\gamma = U$ and $U^\delta = Cl(U)$. Let $A = \{a, b, c\}$, $B = \{a, b\}$, $C = \{a\}$ then A and B are $(\gamma_{id}, \delta_{cl})$ -gc. But C is not $(\gamma_{id}, \delta_{cl})$ -gc.

Theorem 2.3. Superset of a (γ, δ) -gc is always a (γ, δ) -gc.

Proof. Let $(X, \tau, I, \gamma, \delta)$ be DOITS and $A \subset B$ where $A \subseteq X$ and $B \subseteq X$; A is (γ, δ) -gc. If possible consider that $B \subseteq U$, $U \in \tau$ but B is not (γ, δ) -gc. Then clearly $U^\gamma \subseteq A^* \subseteq B^*$ but $B^* \not\subseteq U^\delta$; which contradicts the fact A is (γ, δ) -gc. Hence B is (γ, δ) -gc.

Remark 2.1. \emptyset is not a (γ, δ) -gc but X not necessarily (γ, δ) -gc.

Proof. Let $\emptyset \subseteq U$, $U \in \tau$. As $\emptyset^* = \emptyset$. If possible consider that \emptyset is a (γ, δ) -gc. Then $U^\gamma \subseteq \emptyset \subseteq U^\delta$. If $U \neq \emptyset$, From definition of γ -operator; $\emptyset \subseteq U \subseteq U^\gamma \subseteq \emptyset$. Hence $U = \emptyset$; a contradiction and so \emptyset is not a (γ, δ) -gc.

It can be easily verified that if $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $I = \{\emptyset, \{c\}\}$, let us define γ, δ operator such that $U^\gamma = U$ and $U^\delta = Cl(U)$ then X is $(\gamma_{id}, \delta_{cl})$ -gc.

But if $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}\}$, $I = \{\emptyset, \{a\}\}$, let us define γ, δ operator such that $U^\gamma = U$ and $U^\delta = Cl(U)$ then X is not $(\gamma_{id}, \delta_{cl})$ -gc.

From Theorem 2.2, it is clear that intersection of any two (γ, δ) -gc of DOITS $(X, \tau, I, \gamma, \delta)$ need not be an (γ, δ) -gc.

Theorem 2.4. Let $(X, \tau, I_1, \gamma, \delta)$ and $(Y, \vartheta, I_2, \zeta, \eta)$ be two DOITS. If A is (γ, δ) -gc and B is (ζ, η) -gc then $(A \times B) \subseteq X \times Y$ is said to be $(\gamma_\delta, \zeta_\eta)$ -gc.

Proof. It is easy to check that $(A \times B)^* = A^* \times B^*$. Hence easy to prove.

Definition 2.2. Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then A is said



to be I^S - g -closed if $A^* \subseteq U \cap S$ whenever $A \subseteq U$ and $U \in \tau$ and $S \subseteq X$.

Definition 2.3. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS and $A \subseteq X$ then A is said to be I - (γ, δ) - gc if -

- (1) A is (γ, δ) - gc .
- (2) A is I - g -closed set.
- (3) $U^\gamma = U^\delta$ whenever $\gamma \neq \delta$

Example 2.3. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $I = \{\emptyset, \{c\}\}$, $A = \{a, b, c\}$ then $U = X$, we define γ, δ such that $U^\gamma = Int(Cl(U))$ and $U^\delta = U$ then A is an I - $(\gamma_{ic}, \delta_{id})$ - gc .

Theorem 2.5. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS and $A, F \subseteq X$. If A is I - (γ, δ) - gc and F is both closed and τ^* -closed then $(A \cap F)$ is I - g -closed set but not necessarily I - (γ, δ) - gc .

Proof. Let $(A \cap F) \subseteq U, U \in \tau$. Then $A \subseteq U \cup (X \setminus F)$. As A is I - g -closed set. $A^* \subseteq U \cup (X \setminus F)$. which implies $A^* \cap F \subseteq U$.

Now $(A \cap F)^* \subseteq A^* \cap F \subseteq U$. Thus $(A \cap F)$ is I - g -closed.

Theorem 2.6. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS and $A, F \subseteq X$. If A is I^S - gc and F is both closed and τ^* -closed. If $(U \cap S)$ is clopen for all $U \in \tau$ and $S \subseteq X$, then $(A \cap F)$ is I^S - gc .

Theorem 2.7. Let A be a subset of (X, τ, I) and γ_1, γ_2 and δ_1, δ_2 are operations and $\gamma_1 \wedge \gamma_2 \neq \delta_1 \wedge \delta_2$ then

(1) If A is both (γ_1, δ_1) - gc and (γ_2, δ_2) - gc then A is $(\gamma_1 \wedge \gamma_2, \delta_1 \wedge \delta_2)$ - gc .

(2) with respect to (1); if γ_1, γ_2 and δ_1, δ_2 are mutually dual then A is I - $(\gamma_1 \wedge \gamma_2, \delta_1 \wedge \delta_2)$ - gc . where $\gamma_1 \wedge \gamma_2$ and $\delta_1 \wedge \delta_2$ are defined by $U^{\gamma_1 \wedge \gamma_2} = U^\gamma \cap U^\delta$ and $U^{\delta_1 \wedge \delta_2} = U^{\delta_1} \cap U^{\delta_2}$ for each $U \in \tau$.

Proof. (1) is easy to prove and so omitted.

(2) Since γ_1, γ_2 and δ_1, δ_2 are mutually dual; hence $U^{\gamma_1 \wedge \gamma_2} = U$ and $U^{\delta_1 \wedge \delta_2} = U$. As it satisfies all three conditions of definition 2.3, hence A is I - $(\gamma_1 \wedge \gamma_2, \delta_1 \wedge \delta_2)$ - gc .

Definition 2.4. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS and $A \subseteq X$. we define (γ, δ) - $cl(A) = \cap \{M | A \subseteq M, M \in gc(X)\}$ and (γ, δ) - $int(A) = \cup \{R | R \subseteq A, R \in go(X)\}$. A is called (γ, δ) -closed if (γ, δ) - $cl(A) = A$.

Let $(X, \tau, I, \gamma, \delta)$ and $(Y, \vartheta, I_2, \zeta, \eta)$ be two DOITS. Let $A \subseteq X, B \subseteq Y$. then (γ, δ) - $cl(A \times B) = \cap \{C \times D | A \times B \subseteq C \times D, C \in gc(X), D \in gc(Y)\}$ and (γ, δ) - $int(A \times B) = \cup \{C \times D | C \times D \subseteq A \times B, C \in go(X), D \in go(Y)\}$. Now we call $A \times B$ as product- (γ, δ) -closed if (γ, δ) - $cl(A \times B) = A \times B$.

Lemma 2.1. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS and $A \subseteq X$. then -

- (1) (γ, δ) - $cl(X \setminus A) = X \setminus (\gamma, \delta)$ - $int(A)$.
- (2) (γ, δ) - $int(X \setminus A) = X \setminus (\gamma, \delta)$ - $cl(A)$.

Lemma 2.2. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS and $A, B \subseteq X$, then-

- (1) $A \subseteq (\gamma, \delta)$ - $cl(A)$
- (2) $A \subseteq B \Rightarrow (\gamma, \delta)$ - $cl(A) \subseteq (\gamma, \delta)$ - $cl(B)$
- (3) (γ, δ) - $cl(A \cup B) = (\gamma, \delta)$ - $cl(A) \cup (\gamma, \delta)$ - $cl(B)$.
- (4) (γ, δ) - $cl(A \cap B) \subseteq (\gamma, \delta)$ - $cl(A) \cap (\gamma, \delta)$ - $cl(B)$

Lemma 2.3. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS and $A, B \subseteq X$, then-

- (1) (γ, δ) - $int(A) \subseteq A$
- (2) $A \subseteq B \Rightarrow (\gamma, \delta)$ - $int(A) \subseteq (\gamma, \delta)$ - $int(B)$
- (3) (γ, δ) - $int(A \cup B) \supseteq (\gamma, \delta)$ - $int(A) \cup (\gamma, \delta)$ - $int(B)$.
- (4) (γ, δ) - $int(A \cap B) = (\gamma, \delta)$ - $int(A) \cap (\gamma, \delta)$ - $int(B)$

Theorem 2.8. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS and $A \subseteq X$. If A is (γ, δ) - gc then (γ, δ) - $cl(A) \setminus A$ contains no non-empty (γ, δ) - go .

Proof. Let $F \neq \emptyset$ and F is (γ, δ) - go such that $F \subseteq (\gamma, \delta)$ - $cl(A) \setminus A$. Then $F \subseteq (X \setminus A)$. Then clearly $F \subseteq (\gamma, \delta)$ - $cl(A) \cap (X \setminus (\gamma, \delta)$ -



$cl(A) = \emptyset$. Hence $F = \emptyset$, a contradiction and hence proved.

Lemma 2.4. Let $(X, \tau, I, \gamma, \delta)$ be a DOITS and $A \subseteq X$. Then following result holds –

(1) $A \subseteq Ker(A)$

(2) $A \subseteq B \Rightarrow Ker(A) \subseteq Ker(B)$.

(3) $x \in Ker(A)$ if and only if $A \cap M \neq \emptyset$ where $x \in M, M \in gc(X)$.

Proof. (3) Let $x \in Ker(A)$. If possible let us consider that $A \cap M = \emptyset$ where $x \in M, M \in gc(X)$. Then $A \subseteq (X \setminus M) \Rightarrow Ker(A) \subseteq Ker(X \setminus M) = X \setminus M$. Thus $x \notin M$, a contradiction and hence $A \cap M \neq \emptyset$.

Conversely, Let $x \notin Ker(A)$ and $A \cap M \neq \emptyset$ where $x \in M, M \in gc(X)$. Then $\exists U \in go(X), x \notin U, A \subseteq U$. Thus $A \cap (X \setminus U) \neq \emptyset$ implies $A \not\subseteq U$, a contradiction. Thus $x \in Ker(A)$.

4. Conclusion

In this paper we investigate a special kind of generalized closed set in general topology; whose fuzzy version may be useful in various aspects.

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