The Portfolio Optimizationfor Commercial Banks under Constant Elasticity of Variance Model

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Abstract

Following the recent happening in the world due to the negative effect of Corona virus on the economies of the nations and financial markets, there is need for commercial banks to develop an investment plan which is robust and take into consideration the volatility of the risky assets during investment. Based on this, we study the optimal investment portfolio strategies for a commercial bank with exponential utility under constant elasticity of variance (CEV) model. A portfolio consisting of one risk free asset (treasury security) and two risky assets (marketable security and a loan) such that the risky assets are modelled by CEV model. By using power transformation, change of Variable approach, we obtain explicit solutions of the optimal investment portfolio strategies and the Value function. Furthermore, we obtained the bank's total assets, deposits and capital with numerical simulations.

Keywords: optimal investment portfolio strategy, constant elasticity of variance, exponential utility, commercial bank, volatility.

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1 Introduction

Financial market is currently facing crisis due to the recent spread of coronavirus all over the world. The commercial banks are one of the financial institutions that are worst hit at this point due to the fact they depend heavily on the return on investments from the financial market, and are in business primarily to make a profit. However, for a commercial bank to make maximal profit, it has topay a critical attention to assets management. This involve two factors; namely, the amount of resources available for investment (capital invested, retained earnings and deposits) and the bank's attitude towards risk and return. They know how to distribute their resources among its assets for optimal returns. One of the toolsused in the banking industry is the theory of optimization. The study of stochastic optimization theory is a very valuable instrument in finance. it is usually used to solve a large range of stochastic optimization problems.

Generally, commercial banks are setup to receive money from their customers and lend money to others. They serve institutions and businesses and are also open to the general public. They fulfil many functions which include receiving deposits from depositors, making payments upon the direction of its depositors, collecting funds from other banks payable to their customers, investing funds in securities for a return, safeguarding money, maintaining and servicing savings and checking accounts of their depositors, maintaining custodial accounts, i.e., accounts controlled by one person but for the benefit of another person and lending money [1].

In banking, a number of authors have used the stochastic optimal control technique to solve optimization problems. Common types of banking optimization problems solved with this method are optimal portfolio selection of the bank's capital and optimal capital adequacy management problems. The optimal selection problem has to do with investing bank funds in assets with the aim of generating optimal fund size, while the optimal capital adequacy management problemsinvolves optimizing the capital adequacy ratios. The Basel Committee on Banking Supervision introduced the capital adequacy ratios as a measure of the financial strength of banks and other financial institutions. Some of the authors who carried out research in this area include; [2],studied a problem related to the optimal risk management of banks in a stochastic dynamic setting. In particular, they studied the solution of an optimal stochastic control problem which minimizes bank market and capital adequacy risks by making choices about security allocation and capital requirements.[3], modelled non-risk-based and risk-based capital adequacy ratios. More specifically, they construct continuous-time stochastic models for the dynamics of the Leverage, Equity and Tier 1 ratios with the aim of deriving the CAR; also,they obtained an optimal asset allocation strategy for the Leverage Ratio on a given time interval. [4], investigate the investment of bank funds in loans and treasuries with the aim of generating an optimal final fund level; In their work,they considered cases where the bank that takes behavioural aspects such as risk and regret into account.[5],used Cox-Huang approach to studycommercial banking problem where the interest rate is of affine structure in a continuous-time. They solved the optimal capital allocation strategy that optimizes the banks total non-risk-weighted assets (TNRWAs) consisting of three assets namely a treasury, a marketable security and a loan. [6], considered asset portfolio and capital adequacy management in banking. they model a Basel III compliant commercial bank that operates in a financial market consisting of a treasury security, a marketable security, and a loan. Also, they considered the risk free interest to be stochastic. [7],investigated optimal portfolio strategy for banks, whose shareholders have a power utility function and considers investment in a bank account, securities, and loans. They derived the solution to their problem by following the dynamic programming approach. [8], studied optimal investment strategy and multiperiod deposit insurance pricing model for commercial banks. They considered investment in one risk free asset and two risky assets comprising of marketable security, and a loan such that the prices of the risky assets are modelled based on geometric Brownian motion.

From the available literature and to the best of my knowledge, no work have been recorded where the risky assets are modelled by constant elasticity of variance model in determining the optimal portfolio strategy for any commercial bank. The constant elasticity of variance model was developed by [9] and is a natural extension geometric Brownian motion. According to [10], the model is capable of capturing the implied volatility skew. A lot of researchers such as [11,12] studied utility maximization under constant elasticity model in DC pension scheme. [13], studied optimal investment and reinsurance problem of utility maximization under CEV model. [10], studied optimal investment problem with taxes, dividend and transaction cost using CEV model and logarithm utility function. [14], studied optimal investment strategy with multiple contributors in a DC pension fund using Legendre transformation method. [15], solved the optimal investment problem for a DC pension plan with default risk and return of premiums clauses; they assumed they stock market price followed CEV model. [16] investigated the effect of additional voluntary contribution on the optimal investment strategy under CEV model; they used the power transformation method in solving their problem.In this paper, we investigate the optimal portfolio strategies for a commercial bank. We consider a bank whose investment is spread across one risk free asset (treasury security) and two risky assets (marketable security, and a loan) where the risky assets are allowed to follow the constant elasticity of variance. We will solve for the bank assets, bank deposits and the bank's capital at time t.

2. Preliminaries

Consider a financial market with portfolio comprising of one risk free asset and one risky asset which is continuously open over a fixed interval $[0, T]$, for $T > 0$representing the expiring date of the investment. Let (Ω, F, P) be a complete probability space where Ω is a real space and P is a probability measure, $\{\mathcal{B}_0(t), \mathcal{B}_1(t), \mathcal{B}_2(t): \mathrm{t} \geq 0\}$ is a standard two dimensional Brownian motion and F is the filtration which represents the information generated by the two Brownian motions. Let$\mathcal{S}_t(t)$ denote the price of the risk free asset at time t and it is modelled as follows

$$\frac{d\mathcal{S}_0(t)}{\mathcal{S}_0(t)} = r(t)dt \qquad \mathcal{S}_0(0) = \mathcal{s}_0 > 0 \qquad (2.1)$$

Where $r > 0$ s and represent the risk free interest rate.

Let $\mathcal{S}_1(t)$and $\mathcal{S}_2(t)$denote the prices of stock and loan respectively and their dynamics are given based on the stochastic differential equationsas follows at $t \geq 0$

$$\frac{d\mathcal{S}_1(t)}{\mathcal{S}_1(t)} = (r+\mu_1)dt + \sigma_1\mathcal{S}_t^{\beta}(t)d\mathcal{B}_0(t) \qquad (2.2)$$

$$\frac{d\mathcal{S}_2(t)}{\mathcal{S}_2(t)} = (r+\mu_2)dt + \sigma_2\mathcal{S}_t^{\beta}(t)d\mathcal{B}_1(t) + \sigma_3\mathcal{S}_t^{\beta}(t)d\mathcal{B}_2(t) \qquad (2.3)$$

Where μ_1, μ_2, σ_1, σ_2 and σ_3 are positive and β represents the elasticity parameter. The loan is to be amortized over a period $[0,\ T]$. $\mathcal{B}_0(t)$, $\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$ relate in such way $d\mathcal{B}_0(t)d\mathcal{B}_1(t) = d\mathcal{B}_0(t)d\mathcal{B}_2(t) = d\mathcal{B}_1(t)d\mathcal{B}_2(t) = 0$.

Let φ be the optimal portfolio investment strategy and we define the utility attained by the investor from a given state a at time t as

$$\mathcal{L}_{\varphi}(t, s_1, s_2, a) = E_{\sup_{\varphi}}\left[U\big(\mathcal{A}(T)\big) \mid \mathcal{S}_1(t) = s_1, \mathcal{S}_2(t) = s_2, \mathcal{A}(t) = a\right], \qquad (2.4)$$

where t is the timeand a is the wealth. The objective here is to determine the optimal portfolio strategy and the optimal value function of the investor given as

$$\varphi^* \quad \text{and } \mathcal{L}(t, s_1, s_2, a) = \sup_{\varphi}\mathcal{L}_{\varphi}(t, s_1, s_2, a) \qquad (2.5)$$

Respectively such that

$$\mathcal{L}_{\varphi^*}(t, s_1, s_2, a) = \mathcal{L}(t, s_1, s_2, a).$$

The value function $\mathcal{L}_{\varphi}(t, s_1, s_2, a)$ can be considered as a kind of utility function. The marginal utility of $\mathcal{L}_{\varphi}(t, s_1, s_2, a)$ is a constant, while the marginal utility of the original utility function $U\big(\mathcal{A}(T)\big)$ decreases to zero as $\mathcal{A}(T)\rightarrow\infty$ Kramkov and Schacher-mayer [18]. .$\mathcal{L}_{\varphi}(t, s_1, s_2, a)$ also inherits the convexity of $U\big(\mathcal{A}(T)\big)$ [17]. More precisely, it is strictly convex for $t < T$ even if $U\big(\mathcal{A}(T)\big)$ is not.

To understand the operation and management of a commercial bank, for a practical problem we study its stylized balance sheet, which records the assets (uses of funds) and liabilities (sources of funds) of the bank [8]. The role of bank capital is to balance the assets and liabilities of the bank. A useful way, for our analysis, of representing the balance sheet of the bank is as follows:

$$R + S + L = D + B + C \tag{2.6}$$

where R, S, L, D, B and C represent the values of reserves, securities, loans, deposits, borrowings and capital respectively. Each of the variables above is regarded as a stochastic process. The definitions of the terms above are can be found in [6, 8]. In order for a commercial bank to make a profit, it is important that the bank manages the asset side of its balance sheet properly.

Let $\mathcal{A}(t)$ represent the bank's total asset at time $t \geq 0$. The dynamics of the bank's total asset at time $t \geq 0$ is given by the stochastic differential equation

$$d\mathcal{A}(t) = \mathcal{A}(t)\left(\varphi_0 \frac{d\mathcal{S}_0(t)}{\mathcal{S}_0(t)} + \varphi_1 \frac{d\mathcal{S}_1(t)}{\mathcal{S}_1(t)} + \varphi_2 \frac{d\mathcal{S}_2(t)}{\mathcal{S}_2(t)}\right) + d\mathcal{C}(t) \qquad \mathcal{A}(0) = 1 \tag{2.7}$$

where $d\mathcal{C}(t)$ denote the rate of influx of the bank's capital whose dynamics is given as

$$d\mathcal{C}(t) = cdt\ , \mathcal{C}(0) > 0 \tag{2.8}$$

Let τ_0, τ_1 and τ_2 represent the amount invested in each of the three assets such that

$$\tau_0 = \mathcal{A}(t)\varphi_0,\ \tau_1 = \mathcal{A}(t)\varphi_1 \text{ and } \tau_2 = \mathcal{A}(t)\varphi_2 \tag{2.9}$$

From (2.8) and (2.9), equation (2.7)

$$d\mathcal{A}(t) = \begin{pmatrix} [r\tau_0 + \tau_1(r+\mu_1) + \tau_2(r+\mu_2) + c]dt \\ +\sigma_1 \mathcal{S}_1^{\beta}(t)d\mathcal{B}_0(t) + \sigma_2 \mathcal{S}_2^{\beta}(t)d\mathcal{B}_1(t) + \sigma_3 \mathcal{S}_2^{\beta}(t)d\mathcal{B}_2(t) \end{pmatrix} \qquad \mathcal{A}(0) = 1 \tag{2.10}$$

Applying the maximum principle to (2.10), the Hamilton-Jacobi-Bellman (HJB) equation associated with (2.10) is given as

$$\left\{\begin{matrix} \mathcal{L}_t + (r+\mu_1)s_1\mathcal{L}_{s_1} + (r+\mu_2)s_2\mathcal{L}_{s_2} + c\mathcal{L}_a \\ +\frac{1}{2}\sigma_1^2 s_1^{2\beta+2}\mathcal{L}_{s_1 s_1} + \frac{1}{2}(\sigma_2^2+\sigma_3^2)s_2^{2\beta+2}\mathcal{L}_{s_2 s_2} \\ +\sup\left\{\begin{matrix} \frac{1}{2}\left[\tau_1^2\sigma_1^2 s_1^{2\beta} + \frac{1}{2}\tau_2^2(\sigma_2^2+\sigma_3^2)s_2^{2\beta}\right]\mathcal{L}_{aa} + [r\tau_0 + \tau_1(r+\mu_1) + \tau_2(r+\mu_2)]\mathcal{L}_a \\ +\tau_1\sigma_1^2 s_1^{2\beta+1}\mathcal{L}_{s_1 a} + \tau_2(\sigma_2^2+\sigma_3^2)s_2^{2\beta+1}\mathcal{L}_{s_2 a} \end{matrix}\right\} \end{matrix}\right\} = 0 \tag{2.11}$$

Differentiating (2.11) with respect to $\tau_1$and$\tau_2$and solving it we have

$$\tau_1 = -\frac{\left[(r+\mu_1)\mathcal{L}_a + \sigma_1^2 s_1^{2\beta+1}\mathcal{L}_{s_1 a}\right]}{\sigma_1^2 s_1^{2\beta}\mathcal{L}_{aa}} \tag{2.12}$$

$$\tau_2 = -\frac{\left[(r+\mu_2)\,\mathcal{L}_a + (\sigma_2^2+\sigma_3^2)\mathcal{s}_2^{2\beta+1}\mathcal{L}_{\mathcal{s}_2 a}\right]}{(\sigma_2^2+\sigma_3^2)\mathcal{s}_2^{2\beta}\,\mathcal{L}_{aa}} \quad (2.13)$$

Substituting (2.12) and (2.13) into (2.11)

$$\begin{bmatrix} \mathcal{L}_t + (r+\mu_1)\mathcal{s}_1\mathcal{L}_{\mathcal{s}_1} + (r+\mu_2)\mathcal{s}_2\mathcal{L}_{\mathcal{s}_2} + [r\tau_0 + c]\mathcal{L}_a + \frac{1}{2}\sigma_1^2\mathcal{s}_1^{2\beta+2}\left[\mathcal{L}_{\mathcal{s}_1\mathcal{s}_1} - \frac{\mathcal{L}_{\mathcal{s}_1 a}^2}{\mathcal{L}_{aa}}\right] \\ +\frac{1}{2}(\sigma_2^2+\sigma_3^2)\mathcal{s}_2^{2\beta+2}\left[\mathcal{L}_{\mathcal{s}_2\mathcal{s}_2} - \frac{\mathcal{L}_{\mathcal{s}_2 a}^2}{\mathcal{L}_{aa}}\right] - (r+\mu_1)\mathcal{s}_1\frac{\mathcal{L}_a\mathcal{L}_{\mathcal{s}_1 a}}{\mathcal{L}_{aa}} + (r+\mu_2)\mathcal{s}_2\frac{\mathcal{L}_a\mathcal{L}_{\mathcal{s}_2 a}}{\mathcal{L}_{aa}} \end{bmatrix} \quad (2.14)$$

Where, $\mathcal{L}(T,\mathcal{s}_1,\mathcal{s}_2,a) = U(a)$ and $U(a)$ is the marginal utility of the bank. Next, we proceed to solve (2.14) for $\mathcal{L}$ considering a bank with exponential utility, then substitute the solution in (2.12) into (2.13)

3. Optimal Investment Portfolio Strategy of a Bank with Exponential Utility Function

Assume the member takes a logarithm utility

$$U(a) = -\frac{1}{k}e^{-ka} \qquad k > 0 \quad (3.1)$$

The absolute risk aversion of a bank with the utility described in (3.1) is constant. We form a solution for (2.14) similar to the one in (Muller, 2018) with the form below:

$$\begin{cases} \mathcal{L}(t,\mathcal{s}_1,\mathcal{s}_2,a) = \frac{1}{k}e^{-ka+m(t,\mathcal{s}_1,\mathcal{s}_2,)} \\ m(T,\mathcal{s}_1,\mathcal{s}_2,) = 0, \end{cases} \quad (3.2)$$

Differentiating

$$\left.\begin{matrix} \mathcal{L}_t = m_t\mathcal{L},\ \mathcal{L}_{\mathcal{s}_1} = m_{\mathcal{s}_1}\mathcal{L}\ ,\ \mathcal{L}_{\mathcal{s}_2} = m_{\mathcal{s}_2}\mathcal{L}, \mathcal{L}_{\mathcal{s}_1\mathcal{s}_1} = (m_{\mathcal{s}_1}^2 + m_{\mathcal{s}_1\mathcal{s}_1})\mathcal{L}, \\ \mathcal{L}_{\mathcal{s}_2\mathcal{s}_2} = \left(m_{\mathcal{s}_2}^2 + m_{\mathcal{s}_2\mathcal{s}_2}\right),\ \mathcal{L}_{a\mathcal{s}_1} = -km_{\mathcal{s}_1}\mathcal{L},\ \ \mathcal{L}_{a\mathcal{s}_2} = -km_{\mathcal{s}_2}\mathcal{L} \\ \mathcal{L}_a = -k\mathcal{L}, \mathcal{L}_{aa} = k^2\mathcal{L} \end{matrix}\right\} \quad (3.3)$$

Substituting (3.3) into (2.14), we have

$$\left[m_t - [r\tau_0 + c]k + \frac{1}{2}\sigma_1^2\mathcal{s}_1^{2\beta+2}m_{\mathcal{s}_1\mathcal{s}_1} + \frac{(r+\mu_1)^2}{2\sigma_1^2\mathcal{s}_1^{2\beta}} + \frac{1}{2}(\sigma_2^2+\sigma_3^2)\mathcal{s}_2^{2\beta+2}m_{\mathcal{s}_2\mathcal{s}_2} + \frac{(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2)\mathcal{s}_1^{2\beta}}\right]\mathcal{L} = 0 \quad (3.4)$$

We assume that since $\mathcal{L} \neq 0$, then

$$\left[m_t - [r\tau_0 + c]k + \frac{1}{2}\sigma_1^2 s_1^{2\beta+2} m_{s_1 s_1} + \frac{(r+\mu_1)^2}{2\sigma_1^2 s_1^{2\beta}} + \frac{1}{2}(\sigma_2^2 + \sigma_3^2) s_2^{2\beta+2} m_{s_2 s_2} + \frac{(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2) s_1^{2\beta}}\right] = 0$$

(3.5)

Taking the boundary condition $m(T, s_1, s_2,) = 0$ into consideration, we find the solution to (3.5) as follows

Proposition 3.1 The solution of equation (3.4) is given as

$$m(t, s_1, s_2,) = f_0 + s_1^{-2\beta} f_1 + s_2^{-2\beta} f_2$$

(3.6)

Where

$$\left[\begin{array}{c} f_0 = [r\tau_0 + c]k(t-T) - \frac{\beta(2\beta+1)}{2}[(r+\mu_1)^2 + (r+\mu_2)^2]\left(t\left[T - \frac{t}{2}\right] - \frac{3T^2}{2}\right) \\ f_1 = \frac{(r+\mu_1)^2}{2\sigma_1^2}(T-t) \\ f_2 = \frac{(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2)}(T-t) \end{array}\right]$$

(3.7)

Proof. Let

$$\begin{cases} m(t, s_1, s_2,) = u(t,, p) + v(t,, q), \quad p = s_1^{-2\beta}, \ q = s_2^{-2\beta} \\ u(T, p) = v(T, q) = 0 \end{cases} \tag{3.8}$$

Then

$$\left.\begin{array}{c} m_t = u_t + v_t, \ m_{s_1} = -2\beta s_1^{-2\beta-1} u_p, m_{s_1 s_1} = 2\beta(2\beta+1) s_1^{-2\beta-2} u_p + 4\beta^2 s_1^{-4\beta-2} u_{pp}, \\ m_{s_2} = -2\beta s_2^{-2\beta-1} v_q, m_{s_2 s_2} = 2\beta(2\beta+1) s_2^{-2\beta-2} v_q + 4\beta^2 s_2^{-4\beta-2} v_{qq} \end{array}\right\} \tag{3.9}$$

Substituting (3.9) into (3.5), we have

$$\left[\begin{array}{c} u_t + v_t - [r\tau_0 + c]k + \frac{1}{2}\sigma_1^2[2\beta(2\beta+1)u_p + 4\beta^2 p u_{pp}] + \frac{p(r+\mu_1)^2}{2\sigma_1^2} \\ + \frac{1}{2}(\sigma_2^2 + \sigma_3^2)[2\beta(2\beta+1)v_q + 4\beta^2 q v_{qq}] + \frac{q(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2)} \end{array}\right] = 0$$

(3.10)

Next we assume a solution for equation (3.10) in the form

$$\begin{cases} u(t,, p) + v(t,, q) = f_0 + p f_1 + q f_2 \\ f_0(T) = f_1(T) = f_2(T) = 0 \end{cases}$$

(3.11)

Differentiating (3.11) with respect to $t,\ p,,q$, we have

$$u_t + v_t = f_{0t} + pf_{1t} + qf_{2t}u_p = f_1\,, u_{pp} = 0\,, v_q = f_2\,, v_{qq} = 0$$

(3.12)

Substituting equation (3.12) into (3.11), we have

$$\begin{bmatrix} f_0 - [r\tau_0 + c]k + \sigma_1^2\beta(2\beta+1)f_1 + (\sigma_2^2 + \sigma_3^2)\beta(2\beta+1)f_2 \\ +p\left(f_1 + \frac{(r+\mu_1)^2}{2\sigma_1^2}f_1\right) + q\left(f_2 + \frac{(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2)}f_2\right) \end{bmatrix} = 0$$

(3.13)

Splitting equation (3.13), we have

$$\begin{cases} f_0 - [r\tau_0 + c]k + \sigma_1^2\beta(2\beta+1)f_1 + (\sigma_2^2 + \sigma_3^2)\beta(2\beta+1)f_2 = 0 \\ f_0(T) = 0 \end{cases}$$

(3.14)

$$\begin{cases} f_1 + \frac{(r+\mu_1)^2}{2\sigma_1^2}f_1 = 0 \\ f_1(T) = 0 \end{cases}$$

(3.15)

$$\begin{cases} f_2 + \frac{(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2)}f_2 = 0 \\ f_2(T) = 0 \end{cases}$$

(3.16)

Solving equation (3.14), (3.15) and (3.16), we have

$$f_0 = [r\tau_0 + c]k(t - T) - \frac{\beta(2\beta+1)}{2}[(r+\mu_1)^2 + (r+\mu_2)^2]\left(t\left[T - \frac{t}{2}\right] - \frac{3T^2}{2}\right)$$

$$f_1 = \frac{(r+\mu_1)^2}{2\sigma_1^2}(T - t)$$

$$f_2 = \frac{(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2)}(T - t)$$

Hence completing the prove.

Proposition 3.2 The optimal value function is given as

$$\mathcal{L}(t,\mathcal{s}_1,\mathcal{s}_2,a)=\frac{1}{k}\exp\left(-ka+\begin{bmatrix}[r\tau_0+c]k(t-T)\\ -\frac{\beta(2\beta+1)}{2}[(r+\mu_1)^2+(r+\mu_2)^2]\left(t\left[T-\frac{t}{2}\right]-\frac{3T^2}{2}\right)\\ +\mathcal{s}_1^{-2\beta}\frac{(r+\mu_1)^2}{2\sigma_1^2}(T-t)+\mathcal{s}_2^{-2\beta}\frac{(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2)}(T-t)\end{bmatrix}\right)$$

(3.17)

ProofFrom proposition 3.1,

$$\begin{bmatrix}m(t,\mathcal{s}_1,\mathcal{s}_2,)=[r\tau_0+c]k(t-T)-\frac{\beta(2\beta+1)}{2}[(r+\mu_1)^2+(r+\mu_2)^2]\left(t\left[T-\frac{t}{2}\right]-\frac{3T^2}{2}\right)\\ +\mathcal{s}_1^{-2\beta}\frac{(r+\mu_1)^2}{2\sigma_1^2}(T-t)+\mathcal{s}_2^{-2\beta}\frac{(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2)}(T-t)\end{bmatrix}$$

(3.18)

Also from equation (3.2), we have

$$\mathcal{L}(t,\mathcal{s}_1,\mathcal{s}_2,a)=\frac{1}{k}e^{-ka+m(t,\mathcal{s}_1,\mathcal{s}_2,)}$$

Substituting equation (3.18) into the above equation we have (3.17) which complete the proof.

Proposition 3.3 The optimal portfolio strategiesfor the three assets are given as

$$\varphi_0=1-\varphi_1-\varphi_2$$

(3.19)

$$\varphi_1=\frac{r+\mu_1}{\mathcal{A}k\sigma_1^2\mathcal{s}_1^{2\beta}}$$

(3.20)

$$\varphi_2=\frac{r+\mu_2}{\mathcal{A}\mathrm{k}(\sigma_2^2+\sigma_3^2)\mathcal{s}_2^{2\beta}}$$

(3.21)

ProofRecall from equation (2.12) and (2.13) that the optimal amount invested in the two risky assets are given as

$$\tau_1=-\frac{\left[(r+\mu_1)\,\mathcal{L}_a+\sigma_1^2\mathcal{s}_1^{2\beta+1}\mathcal{L}_{\mathcal{s}_1a}\right]}{\sigma_1^2\mathcal{s}_1^{2\beta}\,\mathcal{L}_{aa}}$$

$$\tau_2=-\frac{\left[(r+\mu_2)\,\mathcal{L}_a+(\sigma_2^2+\sigma_3^2)\mathcal{s}_2^{2\beta+1}\mathcal{L}_{\mathcal{s}_2a}\right]}{(\sigma_2^2+\sigma_3^2)\mathcal{s}_2^{2\beta}\,\mathcal{L}_{aa}}$$

Also from proposition 3.2, we have

$$\mathcal{L}_a=-k\mathcal{L},\mathcal{L}_{aa}=k^2\mathcal{L},\ \mathcal{L}_{a\mathcal{s}_1}=0,\ \mathcal{L}_{a\mathcal{s}_2}=0$$

(3.22)

Putting equation (3.22) into (2.12) and (2,13), we obtain

$$\tau_1 = \frac{r+\mu_1}{k\sigma_1^2 s_1^{2\beta}}$$

(3.23)

$$\tau_2 = \frac{r+\mu_2}{k(\sigma_2^2+\sigma_3^2)s_2^{2\beta}}$$

(3.24)

From equation (2.9)

$$\varphi_1 = \frac{\tau_1}{\mathcal{A}}, \ \varphi_2 = \frac{\tau_2}{\mathcal{A}}$$

(3.25)

Substituting (3.25) into (3.23) and (3.24), we have

$$\varphi_1 = \frac{r+\mu_1}{\mathcal{A}k\sigma_1^2 s_1^{2\beta}} \ , \ \varphi_2 = \frac{r+\mu_2}{\mathcal{A}(\sigma_2^2+\sigma_3^2)s_2^{2\beta}}$$

Which complete the proof.

Remark 3.1 If the elasticity parameter $\beta = 0$, equation (3.17), (3.19), (3.20) and (3.21) reduce to that of (Muller, 2018) as follows

$$\mathcal{L}(t, s_1, s_2, a) = \frac{1}{k} e^{-ka+\left[\begin{array}{c}[r\tau_0+c]k(t-T) \\ +\frac{(r+\mu_1)^2}{2\sigma_1^2}(T-t)+\frac{(r+\mu_2)^2}{2(\sigma_2^2+\sigma_3^2)}(T-t)\end{array}\right]}$$

$$\varphi_0 = 1-\varphi_1-\varphi_2 \qquad , \varphi_1 = \frac{r+\mu_1}{\mathcal{A}k\sigma_1^2} \qquad \text{and } \varphi_2 = \frac{r+\mu_2}{\mathcal{A}k(\sigma_2^2+\sigma_3^2)}$$

Let $\mathcal{D}(t)$represent the bank's total liabilities which is given in [8] by the stochastic differential equation

$$\begin{cases} d\mathcal{D}(t) = \mu_3 dt + \sigma_4 d\mathcal{B}_3(t) \\ \qquad \mathcal{D}(0) = 0.8 \end{cases}$$

(3.26)

Assume $\mathcal{K}(t)$ is the bank's capital at time t, then $\mathcal{K}(t)$ is the difference between the bank's total assets and the bank's total liabilities at time t and is given as

$$\mathcal{K}(t) = \mathcal{A}(t) - \mathcal{D}(t)$$

(3.27)

From (2.27), we have

$$\begin{cases} d\mathcal{K}(t) = d\mathcal{A}(t) - d\mathcal{D}(t) \\ \mathcal{K}(0) = 0.2 \end{cases} \quad (3.28)$$

Solving equation (2.26), we have

$$\mathcal{D}(t) = 0.8 + \left(\mu_3 - \frac{1}{2}\sigma_4^2\right)t + \sigma_4 \mathcal{B}_3(t) \quad (3.29)$$

Solving (2.10) we have

$$\mathcal{A}(t) = 1 + (r+c)t + \frac{\mu_1(r+\mu_1)}{k\sigma_1^2}\int_0^t s_1^{-2\beta}(b)db + \frac{\mu_2(r+\mu_2)}{k(\sigma_2^2+\sigma_3^2)}\int_0^t s_2^{-2\beta}(b)db$$

Solving equation (3.28), we have

$$\mathcal{K}(t) = \begin{pmatrix} 0.2 + (r+c)t + \left(\mu_3 - \frac{1}{2}\sigma_4^2\right)t + \sigma_4\mathcal{B}_3(t) \\ + \frac{\mu_1(r+\mu_1)}{k\sigma_1^2}\int_0^t s_1^{-2\beta}(b)db + \frac{\mu_2(r+\mu_2)}{k(\sigma_2^2+\sigma_3^2)}\int_0^t s_2^{-2\beta}(b)db \end{pmatrix}$$

5. Numerical Simulations

In this section, we give numerical simulation of the optimal portfolio strategies when $\beta = 0$. To achieve this, the following data as in [8] unless otherwise stated. $r = 0.065$, $\mu_1 = 0.035$, $\sigma_1 = 0.08$, $\mu_2 = 0.045$, $\sigma_2 = 0.095$, $\sigma_3 = 0.065$, $\mu_3 = 0.12$, $\sigma_4 = 0.15$, $k = 25$, $c = 0.0145$, $T = 10$

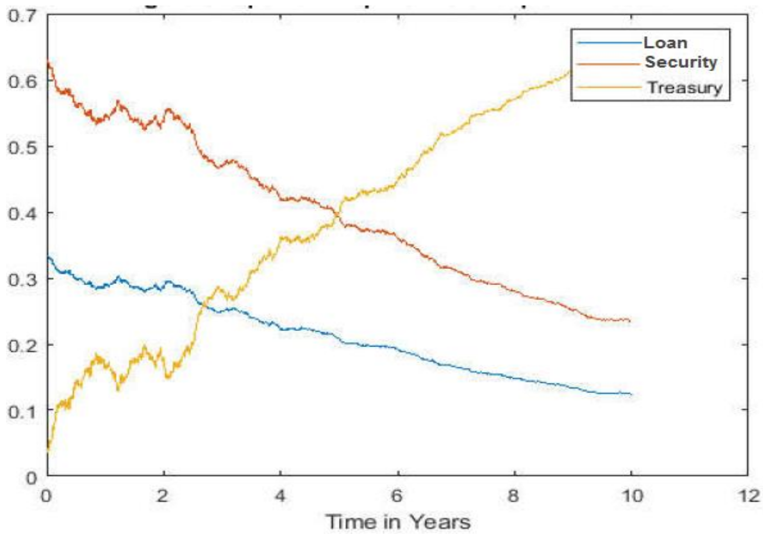

Fig. 1:Evolution of the optimal portfolio strategies with time

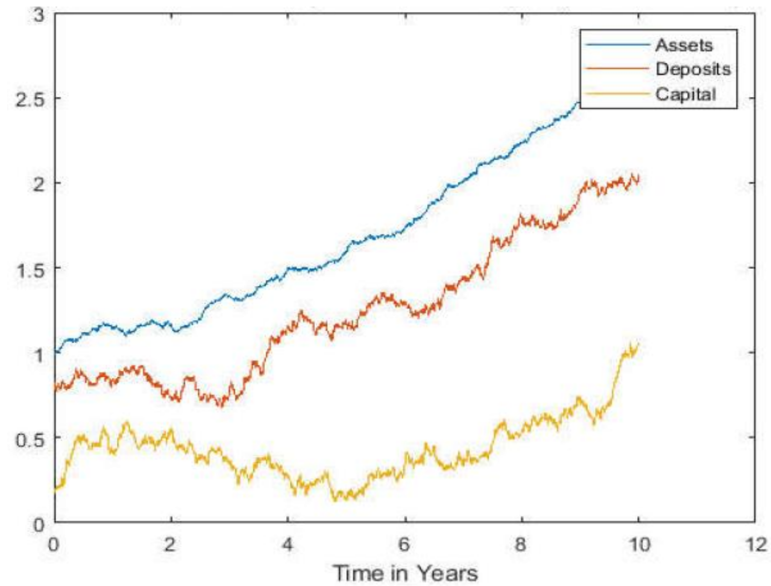

Fig. 2: Time evolution of Asset, Deposits and Capital

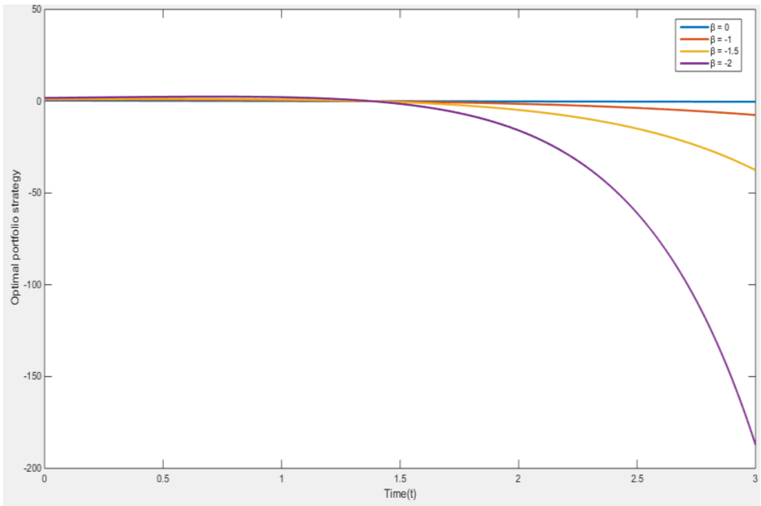

Fig 3: Time evolution of optimal portfolio strategy with different β

6. Discussion

We observed that over time, the optimal portfolio investment strategies of the risky assets decrease continuously while that of the risk free asset increase continuously. Also we observed that the proportion of the bank's wealth invested in loan is smaller compared to that of loan; this could be due to the fact that loan is more risky than stock. This result is consistent with [5-8]. From figure 2, we observed that the based on the investment strategy implore by the bank, the bank's asset is higher than that of its liability showing that the bank makes profit other than loss. Figure 3, present a simulation of the optimal portfolio strategy against time with different values of the elasticity parameter β. The graph shows that as the elasticity parameter decreases, the bank is more scared to invest in risky asset as expiration date approaches. Furthermore, we observed a farther decrease when $\beta = -2$, showing how volatile the risky asset can be hence discouraging for investors with high risk aversion coefficient but when $\beta = 0$, the decline is almost unnoticeable, making the risky asset looks not volatile. This is shows that geometric Brownian motions i.e when $\beta = 0$ the bank may be led astray while taking investment decisions especially in times like this.

7. Conclusion

In this paper, we extended the work of [8] by studying the optimal portfolio investment strategy for a commercial bank with exponential utility under constant elasticity of variance (CEV) model. In doing this, we considered a portfolio consisting of one risk free asset (treasury security) and two risky assets (marketable security and a loan) such that the risky assets are modelled by CEV model. By using power transformation, change of Variable approach, we obtain explicit solutions of the optimal investment portfolio strategies and the Value function. Furthermore, we observed that our result generalizes the result in [8].

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