Simulation of Partial differential Equation by Fourier Transformation

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Abstract
The Fourier transform is a tool for mathematics, electrical and other Engineering sciences. The Fourier transform is beneficial in differential equations because it can transform them into equations which are easier to solve. In addition, many transformations can be made simply by applying predefined formulas to the problems of interest. The aim of this paper is study about Fourier transform and its application in some Partial Differential Equation. We discussed some different types of theorems with their appropriate proofs, different types and interesting definitions as for instance the Fourier transform and applications of the Fourier transform in Partial Differential Equation. One of applications of Fourier Transformation in solving Partial differential equation (PDE) such as heat equation, wave equation is discussed in the subsequent topic. In this paper, we shall study basic concepts, facts and techniques in connection with Fourier Transform in partial differential equation PDE.

Keyword: Fourier Transform, Partial differential Equation, Linear form

1. Introduction

Around 1805, Carl Friedrich Gauss invented a revolutionary technique for efficiently computing the coefficients of what is now called 1 a discrete Fourier series. Unfortunately, Gauss never published his work and it was lost for over one hundred years. During the rest of the nineteenth century, variations of the technique were independently discovered several more times, but never appreciated. In the early twentieth century, Carl Runge derived an algorithm similar to that of Gauss that could compute the coefficients on an input with size equal to a power of two and was later generalized to powers of three. According to Pierre Duhamel and M. Hollmann this technique was widely known and used in the 1940's. However, afterWorldWar II, Runge's work appeared to have been forgotten for an unknown reason. Then in 1965, J. W. Cooley and J. W. Tukey published a short five page paper based on some other works of the early twentieth century which again introduced the technique which is now known as the "Fast Fourier Transform." This time, however, the technique could be implemented on a new invention called a computer and could compute the coefficients of a discrete Fourier series faster than many ever thought possible. Since the publication of the Cooley-Tukey paper, engineers have found many applications for the algorithm. Over 2,000 additional papers have been published on the topic. The Fast Fourier Transform (FFT) has become one of the most important techniques in the field of Electrical Engineering.

Discrete Fourier transform or *DFT* is one of the most important problems in applied computer science, with applications in many areas of scientific computing and engineering.[3]

The first *fast Fourier transform* or *FFT* was invented by Carl Friedrich Gauss in 1805 (even predating Fourier's work on harmonic analysis by two years) and reinvented by James W. Cooley and John W. Tukey in 1965. The name FFT is used today to refer to any "fast" method of computing the DFT, usually O (N log$N$). Sometimes the term is used more specifically to refer to some version of the Cooley-Tukey algorithm, typically for input sizes which are powers of 2. A good theoretical survey of general FFTs can Be many paper .However, the paper only goes so far as to derive equations describing the recursive structures used to compute the DFT. It does not include the kind of detailed description of the algorithms needed to write a fast implementation in practice. Charles Van Loan's monograph provides a beautiful and coherent treatment of several FFT algorithms, describing them all in terms of matrix algebra. This book also includes pseudo-Mat Lab of the algorithms, which is quite useful even though the comments about hardware are now a bit dated. To the best of my knowledge it is the only treatise which covers the relevant correctness proofs for all the major FFT frameworks and connects them with pseudo-code, all in a single monograph.

The Fastest Fourier Transform in the West (FFTW) is a highly general FFT implementation for general-purpose CPUs. It can efficiently compute FFTs of any size as well as multi-dimensional FFTs. It also has special support for the case where the input consists of real numbers.

The Discrete Fourier Transform (DFT) is one of the most widely used digital signal processing (DSP) algorithms. DFTs are almost never computed directly, but instead are calculated using the Fast Fourier Transform (FFT), which comprises a collection of algorithms that efficiently calculate the DFT of a sequence. The number of applications for FFTs continues to grow and includes such diverse areas as: communications, signal processing, instrumentation, biomedical engineering, Sonics and acoustics, numerical methods, and applied mechanics.

Applications of Fourier Transformation in solving Partial differential equation (PDE) such as heat equation, wave equation are discussed in the subsequent topic. In this paper, we shall study basic concepts, facts and techniques in connection with Fourier Transform in partial differential equation PDE.

The Fourier Transform

Translation from Fourier integral to Fourier Transform

Our starting for the development of Fourier transform point is the Fourier integral formula

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \quad , \qquad \text{(1a)}$$

Where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\, dx$ (1b)

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\, dx$$

Just as we can express the Fourier series in a complex exponential form, we can express the Fourier integral in complex exponential form. To obtain that, we put (1b) in to (1a). [First we change the dummy integration variable x in (1b) to u, say to avoid confusing that variable with the x occurring (1a), which denote the fixed point at which $f(x)$ being computed.] Thus,[1]

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u)[\cos\omega u\cos\omega x + \sin\omega u\sin\omega x]dud\omega$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} f(u)[cos\omega ucos\omega x + sin\omega usin\omega x]du\right\} d\omega$$

$$= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u)cos\omega(u-x)dud\omega \qquad (2)$$

Since, $\cos(u-x) = cosucosx + sinusinx.$

To introduce complex exponentials, when we express (2) as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u)\frac{e^{i\omega(u-x)} + e^{-i\omega(u-x)}}{2} dud\omega$$

$$= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u)\, e^{i\omega(u-u)} dud\omega + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u) e^{-i\omega(u-x)}\, dud\omega$$

To combine the two terms on the right hand side, let us change the dummy integration variable from ω to $--\omega$ in the first, thus,

$$f(x) = \frac{1}{2\pi}\int_0^{-\infty} \int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}\, du(-d\omega) + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}\, dud\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} \int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}\, dud\omega + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}\, dud\omega,$$

So $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} . \int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}\, dud\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du\right]e^{i\omega x}\, d\omega$(3)

The latter can be split apart as

$f(x) = a\int_{-\infty}^{\infty} c(\omega)\, e^{i\omega x} d\omega$(4a)

And $c(\omega) = b\int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}du$, (4b)

Where the constants a and b are such that $ab = \frac{1}{2\pi}$,we can make (4) resemble more closely by choosing

$a = 1\ and\ b = \frac{1}{2\pi}$. But we will choose $a = \frac{1}{2\pi}\ and\ \ b = 1$.

There is no longer need to distinguish x and u because the x's is confined to (4a), and $u's\ to\ (4b)$.

To minimize nomenclature and to mimic the form of (1), we write $f(x)$ and $c(\omega)$ in the following form:

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\omega)e^{i\omega x}\, d\omega$

(5a)

$c(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx$(5b)

Rather than thinking of (5a) as the Fourier integral of $f\ and\ (5b) as$ giving (give or take factor of $\frac{1}{2\pi}$) the Fourier coefficients $C(\omega)$, we can think of (5a,b) as a transform pair:(5b)defines the Fourier transform $c(\omega)$ of the given function $f(x)$,and (5a) is called the inversion formula because putting $c(\omega)$ in and integrating gives as back f(x). We use the notation $\mathcal{F}(\omega)$, in a place of $c(\omega)$, for the transform so write (5) in the final form as [2]

$F\{f(x)\} = \mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$(6a)

And the inverse Fourier transform as

$F^{-1}\{\mathcal{F}(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{i\omega x}\, d\omega$ (6b)

Thus, the Fourier transform and inversion formulas are not mysterious [1].

Now if we factor out $\frac{1}{2\pi}$ in the complex Fourier integral and put inside the integral we can write the complex Fourier integral as $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\right]d\omega$ (*)

Thus the term in the large parenthesis of (*) can be taken as the Fourier transform. Therefore the Fourier transform of a function f is given by

$\mathcal{F}\,(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$

And the inversion formula is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}(\omega)e^{i\omega x}\, d\omega$$

Example: find the Fourier transform of

$$q(x) = \begin{cases} e^{iax}, 0 < x < 1 \\ 0, \quad otherwise \end{cases}$$

Solution: let $Q(\omega)$ denotes the Fourier transformation of q. Then the Fourier transformation of the given function is evaluated as follows.

$$Q(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(x) e^{-i\omega x} \, dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} q(x) e^{-i\omega x} dx + \int_{0}^{\infty} q(x) e^{-i\omega x} dx \right]$$

We now the first integral $\int_{-\infty}^{\infty} q(x) e^{-i\omega x} dx$ equals to zero because $q(x) = 0$ for all values of x outside the interval$(0, 1)$.Thus we have

$$Q(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} q(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{iax} e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-i(\omega - a)x} dx$$

Integrate by substitution.

Let$u = -i(\omega - a)x \;\rightarrow\; \frac{du}{dx} = -i(\omega - a) \;\rightarrow\; dx = \frac{du}{-i(\omega - a)} = \frac{-du}{i(\omega - a)}$.

Using this result we obtain that

$$Q(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-i(\omega - a)x} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{u} \left(\frac{-du}{i(\omega - a)} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-i(\omega - a)x}}{i(\omega - a)} \right] \Big|_{0}^{1} = \frac{i}{\sqrt{2\pi}} \left[\frac{e^{-i(\omega - a)x}}{(\omega - a)} \right] \Big|_{0}^{1}$$

$$= \frac{i}{\sqrt{2\pi}} \left[\frac{e^{-i(\omega - a)} - 1}{(\omega - a)} \right]$$

$$= \frac{i}{\sqrt{2\pi}} \left(\frac{1 - e^{-i(\omega - a)}}{a - \omega} \right)$$

As the numerator of the Fourier transform is bounded, the denominator causes the transformation to vanish as$|\omega| \rightarrow \infty$. This example shows that a complex function can also have a Fourier transform and, in general that the transform will be complex.[3]

2. **The Fourier Cosine and sine Transform**

Integrals transform in the form of an integral that produces from given function new function depending on different variable. These transformations are of interest mainly as tools for solving ODE and PDE.

2.1. **The Fourier Cosine Transform**

Definition: Suppose the Fourier integral of a piecewise smooth, integrable &absolutely integrable function f is given by

$f(x) = \int_{0}^{\infty} [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)] d\omega$

Where $A(\omega) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \cos(\omega x) \, dx$ and $B(\omega) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \sin(\omega x) \, dx$

If the function is even, then Fourier integral is the Fourier cosine integral. That is,

$f(x) = \int_{0}^{\infty} A(\omega) \cos(\omega x) \, d\omega$ where

$A(\omega) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \cos(\omega x) dx$ (1)

If we set$A(\omega) = \frac{1}{\sqrt{2\pi}} \mathcal{F}_c(\omega)$ where c suggests "cosine", then from equation(1) we get

$\mathcal{F}_c(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)dx$ (2)

Formula (2) gives us from $f(x)$to a new function $\mathcal{F}_c(\omega)$ called the FourierCosine transform of $f(x)$.

And from $f(x) = \int_0^\infty A(\omega)\cos(\omega)d\omega$ we get $f(x) = \frac{1}{2\pi}\int_0^\infty \cos(\omega x)d\omega$.

This is a function that takes us back from $\mathcal{F}c$ to $f(x)$ and it is known as the inverseFourier cosines transform.[4]

Example: Find the Fourier transformation of e^{-ax} when $a > 0$

Solution

$$\mathcal{F}_c\{e^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ax}\cos\omega x\, dx$$

$$e^{i\omega x} = \cos\omega x + i\sin\omega x$$

$$\mathcal{F}_c\{e^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ax}\cos\omega x\, dx = Re\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ax}e^{i\omega x}dx$$

$$= \frac{1}{\sqrt{2\pi}}Re\int_0^\infty e^{(i\omega-a)x}\,dx$$

We can find the value of $\int_0^\infty e^{(i\omega-a)x}\,dx$ by applying the technique of integration by substitution as follows.

$$u = (i\omega - a)x \to \frac{du}{dx} = i\omega - a$$

$$\to \frac{du}{i\omega - a} = dx$$

Thus, $\int_0^\infty e^{(i\omega-a)x}\,dx = \int_0^\infty e^u \frac{du}{i\omega-a} = \frac{1}{a-i\omega}$

The result is obtained by considering that $e^{(i\omega-a)x} \to 0$as $x \to \infty$ and $e^{(i\omega-a)x} \to 1$as $x \to 0$.Therefore $\mathcal{F}_c\{e^{-ax}\} = \frac{1}{\sqrt{2\pi}}Re\{\int_0^\infty e^{(i\omega-a)x}\,dx\} = \frac{1}{\sqrt{2\pi}}Re\left\{\frac{1}{a-i\omega}\right\}$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{a}{\omega^2 + a^2}\right)$$

2.2. **The Fourier Sine Transform**

Definition: Suppose that Fourier integral of a piecewise smooth, integrable & absolutely integrable function $f$is given by

$$f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega.$$

Where

$$A(\omega) = \frac{2}{\pi}\int_0^\infty \cos(\omega x)dx$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty \sin(\omega x)dx.$$

If the function is odd, then the Fourier integral is a Fourier cosine integral. That is $f(x) = \int_0^\infty B(\omega)\sin(\omega x)d\omega$ .Where

$B(\omega) = \frac{2}{\pi}\int_0^\infty \sin(\omega x)$ (3)

If we set $B(\omega)=\frac{1}{\sqrt{2\pi}}\mathcal{F}_s(\omega)$ where s suggests “sine”. Then from equation (3)we have

$$\mathcal{F}_S = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\sin(\omega x)dx$$

(4)

Formula (4) gives form $f(x)$to a new function $\mathcal{F}_s$ which is called as FourierSine transform of$f(x)$.From

$$f(x) = \int_0^\infty B(\omega)\sin(\omega x)dx$$

We get a function that takes us back from $\mathcal{F}_s$ to $f(x)$and it is known as the inverse Fourier Sine Transform.

Example: Find the Fourier sine transform of $f(x) = e^{-ax}$,$a > 0$

Solution

$$\mathcal{F}_s\{e^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ax}\sin\omega x dx$$

$$e^{i\omega x} = \cos\omega x + i\sin\omega x$$

$$\mathcal{F}_s\{e^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ax}\sin\omega x dx = Im\frac{1}{\sqrt{2\pi}}\left\{\int_0^\infty e^{-ax}e^{i\omega x}\,dx\right\}$$

$$= Im\frac{1}{\sqrt{2\pi}}\left\{\int_0^\infty e^{(i\omega-a)x}dx\right\}$$

By integrating by substitution technique we have

$$\int_0^\infty e^{(i\omega-a)x}\,dx = \frac{1}{a-i\omega}$$

Therefore

$$\mathcal{F}_s\{e^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ax}\sin\omega x dx = Im\frac{1}{\sqrt{2\pi}}\left\{\int_0^\infty e^{-ax}e^{i\omega x}\,dx\right\}$$

$$= Im\frac{1}{\sqrt{2\pi}}\left\{\int_0^\infty e^{(i\omega-a)x}dx\right\}$$

$$= Im\frac{1}{\sqrt{2\pi}}\left\{\frac{1}{a-i\omega}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{\omega}{\omega^2+a^2}\right)$$

Properties of Fourier transform

The Fourier transform has basic properties that can be applied for finding Fourier transform of such functions that require further techniques rather than the definition of the Fourier transform. The most common properties include:[5]

2.3. **Linearity property of the Fourier Transform**

Let$a$ and b be any constants, then for any functions f and g where f and$g$arepiecewise smooth, integrable & absolutely integrable functions, Fourier Transform of linear combination of $f$and$g$ is the linear combination of the

Fourier transforms of functions$f$and$g$.That is

$\mathcal{F}\{af(u)+bg(u)\}=a\mathcal{F}\{f(u)\}+b\mathcal{F}\{g(u)\}$

The proof for this property is simple and it is obtained by using the definition of the Fourier transform of a function.

Proof: $\mathcal{F}\{af(u)+bg(u)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(u)+bg(u)]e^{-i\omega u}du$

$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(u)e^{-i\omega u}+bg(u)e^{-i\omega u}]du$

$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}af(u)e^{-i\omega u}du+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}bg(u)\,e^{-i\omega u}du$

$=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)e^{-i\omega u}du+\frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(u)\,e^{-i\omega u}du$

Thus by definition of the Fourier transform

$= a\mathcal{F}\{f\}+b\mathcal{F}\{g\}$

2.4. **Shift properties the Fourier Transform**

There are two shift properties of Fourier Transform:

(I) Time shift property:

$\mathcal{F}\{(u-u_0)\}(\omega)=e^{-i\omega u_0}\mathcal{F}(\omega)$

(II) Frequency shift property:

$\mathcal{F}\{e^{i\omega_0 u}f(u)=\mathcal{F}(\omega-\omega_0).$

Here u_0 and ω_0 are constants. In words shifting a function in one domain corresponds to a multiplication by a complex exponential function in theother domain.[6]

Proof: Time shift property

$\mathcal{F}\{f(u-u_0)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u-u_0)e^{-i\omega u}\,du.$

If we multiply the right hand side by,$e^{i\omega u_0}\,e^{-i\omega u_0}=1$,then we obtain that:

$$\mathcal{F}\{f(u-u_0)\}=\frac{e^{-i\omega u_0}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u-u_0)e^{-i\omega u}du$$

$$=e^{-i\omega u_0}\mathcal{F}(\omega).$$

The inverse version of the time shift property of the Fourier transform isgiven by:

$\mathcal{F}^{-1}\left\{e^{-i\omega_0 u}\mathcal{F}(\omega)\right\}(u)=f(u-u_0)$

Proof:Frequency shift property:

$$\mathcal{F}\left\{e^{i\omega_0 u}f(u)\right\}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\omega_0 u}f(u)e^{-i\omega u}du$$

$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i(\omega-\omega_0)}\,f(u)du$

$$=\mathcal{F}(\omega-\omega_0).$$

The inverse version frequency shift property of the Fourier transform hasthe form:

$\mathcal{F}^{-1}\{\mathcal{F}(\omega-\omega_0)\}=e^{i\omega_0}f(u)$

2.5. The Fourier transform of the Derivative of $f(x)$

2.6. Let $f(x)$be continuous on the x- axis and$f(x)\to 0$ as$|x|\to\infty$. Furthermore, let $f'(x)$ be absolutely integrable on the$x$- axis. Then

$\mathcal{F}\{f'(x)\} = \mathcal{F}\left\{\frac{df}{dx}\right\} = i\omega\mathcal{F}\{f(x)\}$

(a)

Proof:

From definition of the Fourier transform we have

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\,du.$$

Integrating by parts, we obtain

$u = e^{-i\omega x} \rightarrow du = -i\omega e^{-i\omega x}$

$dv = f'(x)du \rightarrow v = f(x).$ Thus, we getresult as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega u}\,du = \frac{1}{\sqrt{2\pi}}[f(x)e^{-i\omega x}]|_{-\infty}^{\infty} + \frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

Since$f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, we have

$\mathcal{F}\{f'(x)\} = 0 + i\omega\mathcal{F}\{f(x)\} = i\omega\mathcal{F}\{f(x)\}$

Two successive application of (a) give

$$\mathcal{F}\{f''\} = \mathcal{F}\left\{\frac{d^2 f}{dx^2}\right\} = (i\omega)^2\mathcal{F}\{f\} = -\omega^2\mathcal{F}\{f(x)\}.$$

Similarly for higher derivatives [2].

2.7. Fourier transforms of partial derivative with respect to x of a function $f(x,t)$ of two independent variables

The Fourier transform with respect to x of a function $f(x,t)$ of two independent variables x and t, denoted by $\mathcal{F}(\omega,t)$, is defined as

$\mathcal{F}\{f(x,t)\} = \mathcal{F}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t)e^{-i\omega x}\,dx.$

(b)

In (b) the variable t is not involved in the integration with respect to x, so it follows that the integral by which $f(x,t)$ is recovered from $\mathcal{F}(\omega,t)$and the transform of partial derivatives of $f(x,t)$ with respect to x obey the same rules as those for the function of a single variable$f(x)$. Thus, the inversion integral is given by

$$f(x,t) = \mathcal{F}^{-1}\{\mathcal{F}(\omega,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}(\omega,t)e^{i\omega x}\,d\omega, \quad (c)$$

and the Fourier transforms of the partial derivatives of $f(x,t)$ with respect to $x$is given by

$$\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}[f(x,t)]\right\} = (i\omega)^n\mathcal{F}(\omega,t)[4].$$

2.8. The Fourier sine and cosine transformation of derivatives

Fourier sine and cosine transforms of derivatives of a function are defined by

1. $\mathcal{F}_s\{f'\} = -\omega\mathcal{F}_c\{f\}$
2. $\mathcal{F}_s\{f''\} = -\frac{2}{\pi}\omega f(0) - \omega^2\mathcal{F}_s\{f\}$
3. $\mathcal{F}_c\{f'\} = -\frac{2}{\pi}f(0) + \omega\,\mathcal{F}_s\{f\}$

4. $\mathcal{F}_c\{f''\} = -\frac{2}{\pi}\omega f(0) - \omega^2 \mathcal{F}_c\{f\}$

The Discrete Fourier Transform

If a function f has fundamental period P, then the complex Fourier series of function f has the following form[7]

$$f(x) = \sum_{n=0}^{\infty} d_n e^{-\frac{2\pi n x}{p}}$$

In which $d_n = \frac{1}{p}\int_0^p f(x)\, e^{-\frac{2\pi n x}{p}}\, dx$, for $n = 0, \pm 1, \pm 2, \ldots$

Under certain conditions on f, this series converges at x to $\frac{f(x+0)+f(x-0)}{2}$

Now to define the discrete Fourier transform, we consider the problem of approximating the coefficients d_n in the complex Fourier series. We begin by sub dividing $[0,p]$ in to N sub intervals of equal length $\frac{p}{N}$ and choosing a point in each interval of subdivision, say

t_α in $\left[\frac{\alpha p}{N}, \frac{(\alpha+1)p}{N}\right]$ for $\alpha = 0,1,2,3,\ldots,N-1$

Approximate d_n by Riemann sum

$$d_n \approx \frac{1}{p}\sum_{\alpha=0}^{N-1} f(x_\alpha) e^{-\frac{2\pi i n x_\alpha}{p}} \frac{p}{N}$$

$$= \frac{1}{N}\sum_{\alpha=0}^{N-1} f(x_\alpha) e^{-\frac{2\pi i n x_\alpha}{p}}$$

This helps us to define the discrete Fourier transform, which acts on a sequence of N given complex numbers and produces an infinite sequence of complex numbers.

Definition: Let $u = [u_\alpha]_{\alpha=0}^{N-1}$ be sequence of N complex numbers, where $N \in \mathbb{P}$. Then N - point discrete Fourier transform of U is the sequence denoted by $\mathcal{D}[U]$ and defined by

$\mathcal{D}[u](n) = \sum_{\alpha=0}^{N-1} u_\alpha e^{-\frac{2\pi i \alpha n}{N}}$ For $n = 0, \pm 1, \pm 2, \ldots$

We can simplify the notation by denoting the N - point discrete Fourier transform of u by U, with lower case for the input sequence and upper case for its discrete transform. In this notation,

$U_n = \sum_{\alpha=0}^{N-1} u_\alpha e^{-\frac{2\pi i \alpha n}{N}}$ For $n = 0, \pm 1, \pm 2, \ldots.$

We also abbreviate the phrase "discrete Fourier transform" to DFT[3].

Example: Let $u = [c]_{\alpha=0}^{N-1}$ be constant sequence with c a given complex number. Then find the N - point DFT of U.

Solution:

$U_n = \sum_{\alpha=0}^{N-1} c e^{-\frac{2\pi i \alpha n}{N}} = c\sum_{\alpha=0}^{N-1} e^{-\frac{2\pi i \alpha n}{N}}$

$U_n = c\sum_{\alpha=0}^{N-1}\left(e^{-\frac{2\pi i n}{N}}\right)^{\alpha}$ This is a finite geometric series. In general, for $|r| < 1$

$$\sum_{\alpha=0}^{N-1} r^\alpha = \frac{1-r^N}{1-r}$$

Then $U_n = \left(\frac{1-\left(e^{-\frac{2\pi i n}{N}}\right)^N}{1-e^{-\frac{2\pi i n}{N}}}\right)c$

$= \left(\frac{1-e^{-2\pi i n}}{1-e^{-\frac{2\pi i n}{N}}}\right)c = 0forn = 0, \pm1, \pm2, \ldots$. This is because for any integer n

$e^{-2\pi i n} = \cos(2\pi n) - i\sin(2\pi n) = 1$

Therefore the N - point DFT of a constant sequence is an infinite sequence of zero.[9]

DERIVATION SOME PARTIAL DIFFERENTIAL EQUATION

In the Fourier series the method of separation of variables was used to obtain solutions of initial and boundary value problems for partial differential equations given over bounded spatial regions. The present topic deals with partial differential equations defined over unbounded spatial regions. The mathematical tools used for solving initial and boundary value problems over unbounded spatial regions are integral transforms: The Fourier transforms (FT), the Fourier sine transforms (FST) and the Fourier cosine transforms (FCT).[8]

HEAT EQUATION

Consider the heat conduction in a rod of constant cross section area

A. Thermal energy density $e(x,t) = \frac{Energy}{Volume}$
B. Heat flux $\emptyset(x,t) =$ the amount of thermal energy flowing across boundaries per unit surface area per time $=\frac{Energy}{Area\ Time}$
C. Heat source $Q(x,t) =$ heat energy per unit volume generated per unit time = $\frac{Energy}{Volume\ Time}$
D. Temperature $U(x,t)$
E. Specific heat $c =$ the heat energy that must be supplied to a unit mass of a substance to rise its temperature one unit $=\frac{Energy}{Mass\ Temperature}$.
F. Mass density $\rho(x) =$ mass per unit volume $=\frac{Mass}{Volume}$.

Conservation of heat energy

Rate of change of heat energy in time = Heat energy flowing across boundaries per unit time + Heat energy generated insider per unit time.

I,Heat energy = Energy density ×Volume = $e(x,t)A\Delta x$.

II, Heat energy flowing across boundaries per unit time = flow-in flux × Area − flow-out flux × Area

$= \emptyset(x,t)A - \emptyset(x+\Delta x,t)$A.

III, Heat energy generated insider per unit time =$Q(x,t)$ × Volume =$Q(x,t)A\Delta x$.

Then

$\frac{\partial}{\partial t}[e(x,t)A\Delta x] = \emptyset(x,t)A - \emptyset(x+\Delta x,t)A + Q(x,t)A\Delta x$.

Dividing it by $A\Delta x$ and letting Δx go to zero give

$\frac{\partial e}{\partial t} = -\frac{\partial \emptyset}{\partial x} + Q.$

Heat energy = $\frac{Energy}{Mass\ Temperature} \times Temperature \times \frac{Mass}{Volume} \times Volume = c(x)u(x,t)\rho A\Delta x.$

So,

$$e(x,t)A\Delta x = c(x)u(x,t)\rho.$$

And then

$e(x,t) = c(x)u(x,t)\rho.$

It then follows from Fourier's law that

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_o\frac{\partial u}{\partial x}\right) + Q.$$

And then the heat equation

$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q}{c\rho}.$

Where $k = \frac{K_o}{c\rho}$ is called the thermal diffusivity.

Consider the heat flow in an infinite rod where the initial temperature is $u(x,0) = f(x)$ we shall prove that if the function $f(x)$ is continuous and either absolutely integrable i.e.,

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$

or bounded (i.e., $|f(x)| \le M \quad \forall x$) then the following IVP problem has a solution $u(x,t)$ which is
continuous throughout the half-plane $t \ge 0, \ge -\infty < x < \infty.$

$$\text{PDE:} \quad u_t(x,t) = \alpha^2 u_{tt}(x,t), \quad -\infty < x < \infty, t > 0,$$

$$\text{IC:} \quad u(x,0) = f(x), \quad -\infty < x < \infty,$$

with$u(x,t), u_x(x,t) \to 0$ as $x \to \pm\infty, t > 0.$

Assume that u and $\frac{\partial u}{\partial x}$ and finite as $|x| \to \infty$ and $u(x,0) = f(x), \quad -\infty < x < \infty,$ where $f(x)$ is piecewise smooth on every finite subinterval and $\int_{-\infty}^{\infty} |f(x)| dx$ is finite.

Define the spatial Fourier transform of $u(x,t)$ to be

$\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx.$

Applying the spatial transform to the differential equation we get

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}(x,t)\,e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c^2\frac{\partial^2 u}{\partial x^2}(x,t)\,e^{-i\omega x}dx,$so that

$$\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx\right] = c^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}(x,t)\,e^{-i\omega x}dx$$

giving $\frac{\partial \hat{u}}{\partial t}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t)$ and hence $\hat{u}(\omega,t) = A(\omega)e^{-c^2\omega^2 t}$, where $A(\omega)$ is some function of ω. Now $u(x,0) = f(x)$ implies that $(\omega,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)\,e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}dx = f(\omega)$,so $A(\omega) = \hat{f}(\omega)$ and $\hat{u}(\omega,t) = \hat{f}(\omega)e^{-c^2\omega^2 t}$. Taking the inverse Fourier transform we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-c^2\omega^2 t}e^{i\omega x}d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{\omega(ix-c^2\omega t)}d\omega$$

The wave equation

We define the wave equation as $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\ t > 0,\ c > 0.$

Assume that u and $\frac{\partial u}{\partial x}$ are finite and $u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(x,0) = g(x), \quad -\infty < x < \infty,$ where $f(x)$ and $g(x)$ are piecewise smooth on every finite subinterval and $\int_{-\infty}^{\infty}|f(x)|dx, \quad \int_{-\infty}^{\infty}|g(x)|\,dx$ are both finite.

Applying the spatial transform to the differential equation we get the following expression

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial t^2}(x,t)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}c^2\frac{\partial^2 u}{\partial t^2}(x,t)e^{-i\omega x}dx$, so that

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx\right] = c^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}(x,t)e^{-i\omega x}dx$$

giving $\frac{\partial^2\hat{u}}{\partial t^2}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t)$ and hence $\frac{\partial^2\hat{u}}{\partial t^2}(\omega,t) + c^2\omega^2\hat{u}(\omega,t) = 0.$

The general solution is

$\hat{u}(\omega,t) = A(\omega)e^{ic\omega t} + B(\omega)e^{-ic\omega t}.$

Now $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ imply that

$$\hat{u}(\omega,0) = \hat{f}(\omega) \text{ and } \frac{\partial\hat{u}}{\partial t}(\omega,0) = \hat{g}(\omega)\text{ , so that}$$

$A(\omega) + B(\omega) = \hat{f}(\omega)$ and $ic\omega\big(A(\omega) - B(\omega)\big) = \hat{g}(\omega).$

Hence $A(\omega) = \frac{1}{2}\left(\hat{f}(\omega) + \frac{\hat{g}(\omega)}{ic\omega}\right)$ and $B(\omega) = \frac{1}{2}\left(\hat{f}(\omega) - \frac{\hat{g}(\omega)}{ic\omega}\right)$ and so

$$u(x,t) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\left(\hat{f}(\omega) + \frac{\hat{g}(\omega)}{ic\omega}\right)e^{ic\omega t}e^{i\omega x} + \left(\hat{f}(\omega) - \frac{\hat{g}(\omega)}{ic\omega}\right)e^{-ic\omega t}e^{i\omega x}\right]d\omega$$

$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega(x+ct)}d\omega + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{i\omega(x-ct)}d\omega + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\omega)\left(\frac{e^{i\omega(x+ct)} + e^{i\omega(x-ct)}}{i\omega}\right)d\omega$$

$=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(v)dv$, which is called D'Alembert's solution.

Conclusion

Throughout the paper, the Fourier transform was introduced and it's most operational properties were established. The number of useful properties of Fourier transform, such as linearity, shifts the transform of derivatives and partial derivatives were considered, and applications were made to functions defined by partial differential equation. Those properties helped us finding the Fourier transform and the inverse Fourier transform of some kinds of functions. For some problems defined on the un-bounded intervals we obtained the Fourier cosine and sine transform and we have applied to solve problems that are showed in the examples.

We used and applied the Fourier transformation method in solving the partial differential equation, such as one dimensional heat equation and wave equations.

Generally the Fourier Transform is a broad topic, of which we have covered only a small fraction. We have taken a rather simple approach, presenting the Fourier transform a mathematical point of view, and providing intuition through examples and applications.

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