

# Numerical Integration of Non-Linear Wave Equations

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## Introduction

Solving partial differential equations is an art that can be difficult, especially when done numerically. There are many factors that must be considered. Our goal is to pick a numerical method that not only makes sense for the equation we are solving, but one that is stable over a given amount of time, one that has a sufficiently high order accuracy, and one that minimizes the time required to compute the solution while keeping the accuracy. Also, boundary conditions must be considered as they can effect a solution drastically and may cause discontinuities.

Applications in numerical partial differential equations fall under gravitational wave research and numerical relativity, to name a few. Current research in the study of gravitational waves includes projects like LIGO (Laser Interferometer Gravitational- Wave Observatory) which are looking to find the existence of gravity waves (see [BW], [W], or [KT] for more information). In finding gravity waves, we require a way to model them in order to understand exactly what is happening. Numerical methods are required to determine whether gravity waves are produced by such events as black holes colliding.

## Nonlinear Systems

In [HKN], the authors present the nonlinear hyperbolic partial differential equation

$$g_{tt} = g_{xx} - \frac{g_t^2}{g}.$$

This equation has nonlinearity similar to that of general relativity, and, in particular, Einstein's equations. By studying this system, we can learn about the strengths and weaknesses of each method as applied to general relativity. The authors of [HKN] show the stability for four different numerical methods, the Iterative Crank-Nicholson, third-order Runge-Kutta, fourth-order Runge- Kutta, and the Courant-Friedrichs-Levy Nonlinear (CFLN) schemes. The CFLN method is similar to Leapfrog, however there is a difference in their evaluation of  $gt$ : In our study, we will look at the three methods we have studied over the past three chapters, with Dirichlet boundary conditions.

We will use the exponential growth function

$$g(x, t) = e^{x+t/\sqrt{2}}$$

as our initial condition. This is a solution of the nonlinear wave equation as

$$g_{tt} = .5g$$

$$g_{xx} - \frac{g_t^2}{g} = g - \frac{.5g^2}{g} = .5g.$$

Unfortunately, by using the exponentially growing solution, we will not be able to consider the types of graphs we have considered previously. The wave, and therefore the norm, grow exponentially, as we expect the error to. Thus, we we will look at plots of the relative error, that is, plots of

$$\frac{\text{calculated value} - \text{actual value}}{\text{actual value}}$$

where the calculated value is the value of the function given by our numerical method and the actual value is

$$g(x, t) = e^{x+t/\sqrt{2}}.$$

As with the wave equation, we will again look at two different systems that are equivalent to our wave. The first system is

$$\begin{cases} g_t = K \\ K_t = w_x - \frac{g_t^2}{g} \\ w_t = K_x \end{cases}$$

and the second system is

$$\begin{cases} g_t = K \\ K_t = g_{xx} - \frac{g_t^2}{g} \end{cases}$$

### Three-Variable System Lax-Wendroff Method

How It Works Recall that Lax-Wendroff approximates  $g$  by the second-order Taylor series, where the derivatives are replaced by their difference equation equivalents. Computing the first- and second-order time derivatives of each variable, we get the difference equations. Since

$$g_t = K,$$

$$g_{tt} = w_x - \frac{K^2}{g},$$

$$K_t = w_x - \frac{K^2}{g},$$

$$K_{tt} = K_{xx} - \frac{2KK_t}{g} + \frac{K^3}{g^2},$$

$$w_t = K_x, \text{ and}$$

$$w_{tt} = K_{xt} = K_{tx} = w_{xx} - \frac{2KK_x}{g} + \frac{Kw}{g^2},$$

we get the difference equations

$$g_i^{n+1} = g_i^n + \tau g_t + \frac{\tau}{2} g_{tt},$$

$$K_i^{n+1} = K_i^n + \tau K_t + \frac{\tau}{2} K_{tt}, \text{ and}$$

$$w_i^{n+1} = w_i^n + \tau w_t + \frac{\tau}{2} w_{tt},$$

where

$$g_t = K_i^n,$$

$$g_{tt} = \frac{w_{i+1}^n - w_{i-1}^n}{2h} - \frac{(K_i^n)^2}{g_i^n},$$

$$K_t = \frac{w_{i+1}^n - w_{i-1}^n}{2h} - \frac{(K_i^n)^2}{g_i^n},$$

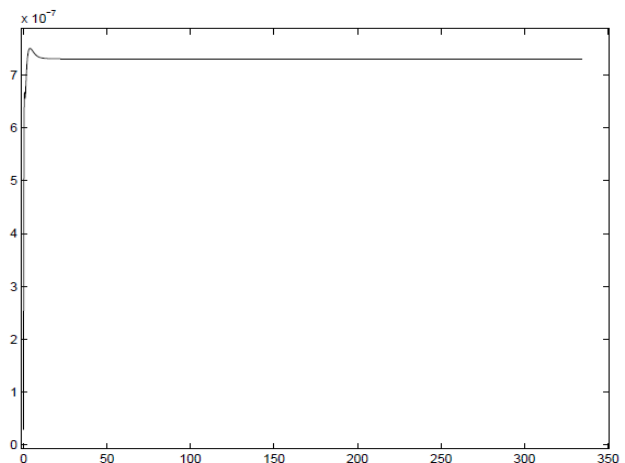
$$K_{tt} = \frac{K_{i+1}^n - K_{i-1}^n}{2h} - \left( \frac{2K_i^n}{g_i^n} \right) \left( \frac{w_{i+1}^n - w_{i-1}^n}{2h} - \frac{(K_i^n)^2}{g_i^n} \right) + \frac{(K_i^n)^3}{(g_i^n)^2},$$

$$w_t = \frac{K_{i+1}^n - K_{i-1}^n}{2h}, \text{ and}$$

$$w_{tt} = \frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{h^2} - \left( \frac{2K_i^n}{g_i^n} \right) \left( \frac{K_{i+1}^n - K_{i-1}^n}{2h} \right) + \frac{K_i^n w_i^n}{(g_i^n)^2}.$$

Solutions Consider Figure 1.

We see for a Courant factor of .8, our method is stable up to around 330 crossing times. At this point, the graph seems to disappear, but not because of instability.



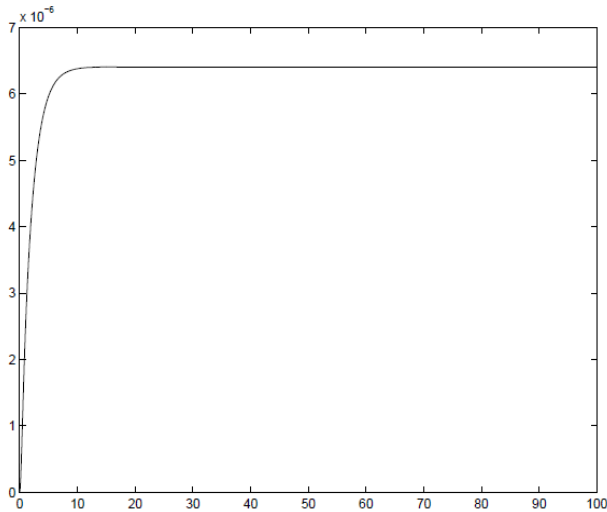


Figure 1: Lax-Wendroff method for Three-Variable Nonlinear Equation (a)  $\alpha = .8$ ; (b)  $\alpha = .1$

This is because the values of the function are becoming too large for the computer to handle. Also notice the magnitude of the error:  $10^{-7}$ . There is very little difference between the two functions! Thus, for all purposes, Lax-Wendroff is stable for our nonlinear system (with a Courant factor of .8). Let's look at how Leapfrog models our solution.

### Leapfrog Method

How It Works Recall from the wave equation that we used half-steps for the Leapfrog method. We will again use half-steps for our nonlinear system. In fact, we will use the same grid system as in the three-variable wave equation. That is, our lattice will be constructed so that we find the values of  $g$  at the lattice points  $(i, n)$ ,

$$(i, n + \frac{1}{2}),$$

the values of  $K$  at the lattice points and the values of  $w$  at the lattice points

$$(i + \frac{1}{2}, n).$$

Thus our difference equations will be

$$g_i^{n+1} = g_i^{n-1} + 2\tau (K_i^{n+.5}),$$

$$K_i^{n+.5} = K_i^{n-.5} + 2\tau \left( \frac{w_{i+.5}^n - w_{i-.5}^n}{h} - \frac{(K_i^{n+.5})^2}{g_i^n} \right), \text{ and}$$

$$w_{i+.5}^{n+1} = w_{i+.5}^{n-1} + 2\tau \left( \frac{K_{i+.5}^n - K_{i-.5}^n}{h} \right).$$

**Solutions** Unfortunately, Leapfrog is not as stable for as long as Lax-Wendroff

Consider Figure 2.

For a Courant factor of .8, our solution blows up after about 50 crossing times.

In plot (b) we see that Leapfrog is stable for much longer. Up to 100 crossing

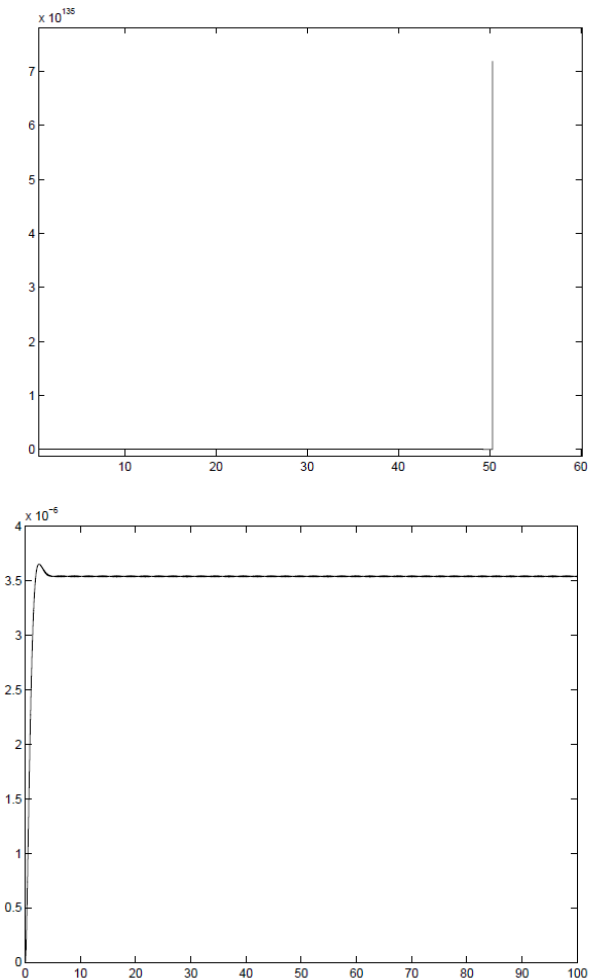


Figure 2: Leapfrog method for Three-Variable Nonlinear Equation (a)  $\alpha = .8$  (b)  $\alpha = .1$  times, Leapfrog models our exponential

function quite accurately with an error on the order of  $10^{-6}$  (not as small as Lax-Wendroff, but still significantly small). We could check beyond 100 crossing times; however, using such a small Courant factor means a much longer time to produce results. In fact, it takes the same amount of time to run a plot for  $\alpha = :1$  for 100 crossing times as it takes to run a plot for  $\alpha = :8$  for 1000 crossing times. Let's see how Crank-Nicholson behaves, using our nonlinear equation.

### Iterative Crank-Nicholson Method

How It Works Recall from before that for the Iterated Crank-Nicholson method, we took each term in the second term on the right hand side of the FTCS method and replaced it by the average of that value and the value of the function at the next time step. Thus our difference equations are

$$g_i^{n+1} = g_i^n + \frac{\tau}{2} (K_i^{n+1} + K_i^n),$$

$$K_i^{n+1} = K_i^n + \frac{\tau}{2} \left( \frac{w_{i+1}^{n+1} - w_{i-1}^{n+1}}{2h} - \frac{(K_i^{n+1})^2}{g_i^{n+1}} + \frac{w_{i+1}^n - w_{i-1}^n}{2h} - \frac{(K_i^n)^2}{g_i^n} \right),$$

$$w_i^{n+1} = w_i^n + \frac{\tau}{2} \left( \frac{K_{i+1}^{n+1} - K_{i-1}^{n+1}}{2} + \frac{K_{i+1}^n - K_{i-1}^n}{2} \right),$$

**Solutions**For a Courant factor of .8, Figure 3(a) shows that Crank-Nicholson blows up at only 4.5 crossing times! This is a very short amount of time, especially compared to our previous two methods. In plot (b), however, we see that Crank-Nicholson gives the same amount of error as Leapfrog. Perhaps the stability condition is stricter than  $\alpha < 1$  for the Leapfrog and Crank-Nicholson methods.

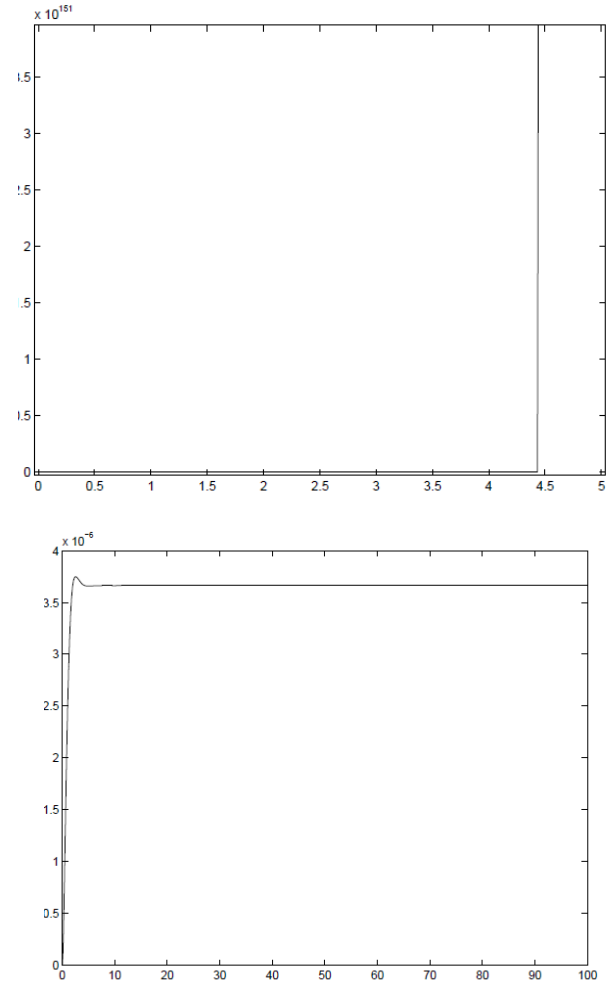


Figure 3: Crank-Nicholson method for Three-Variable Nonlinear Equation (a)  $\alpha = :8$ ; (b)  $\alpha = :1$

Let's see how the two-variable system shapes up.

### Two-Variable System

#### Lax-Wendroff Method

How It Works Recall that Lax-Wendroff approximates  $g$  by the second-order Taylor series, where the derivatives are replaced by there difference equation equivalents. Computing the first- and second-order time derivatives of each variable, we get the difference equations

$$g_i^{n+1} = g_i^n + \tau g_t + \frac{\tau}{2} g_{tt}, \text{ and}$$

$$K_i^{n+1} = K_i^n + \tau K_t + \frac{\tau}{2} K_{tt},$$

Where

$$g_t = K_i^n,$$

$$g_{tt} = \frac{g_{i+1}^n - 2g_i^n + g_{i-1}^n}{h^2} - \frac{(K_i^n)^2}{g_i^n},$$

$$K_t = \frac{g_{i+1}^n - 2g_i^n + g_{i-1}^n}{h^2} - \frac{(K_i^n)^2}{g_i^n},$$

$$K_{tt} = \frac{K_{i+1}^n - K_{i-1}^n}{2h} - \left( \frac{2K_i^n}{g_i^n} \right) \left( \frac{g_{i+1}^n - 2g_i^n + g_{i-1}^n}{h^2} - \frac{(K_i^n)^2}{g_i^n} \right) + \frac{(K_i^n)^3}{(g_i^n)^2}.$$

**Solutions** Compare Figure 4 below to Figure 1 above.

Interestingly, although the three-variable system is stable for higher Courant factors, the relative error for  $\alpha = :1$  is smaller for the two-variable system. In

Figure 55(a), we see that the solution blows up before .5 crossing times whereas in Figure 52, the solution was steady up to 330 crossing times. Of course, as with the

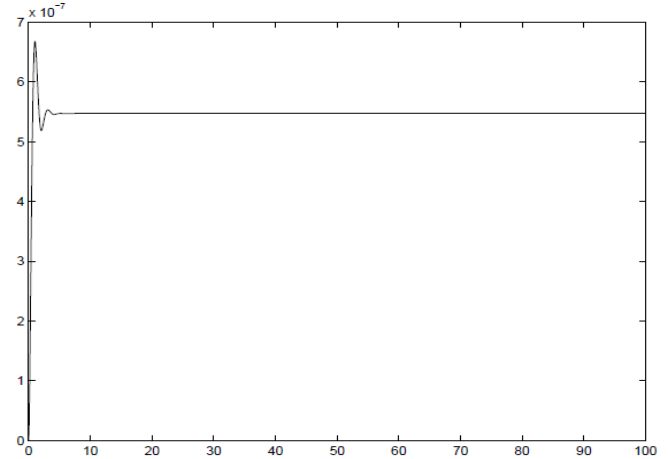
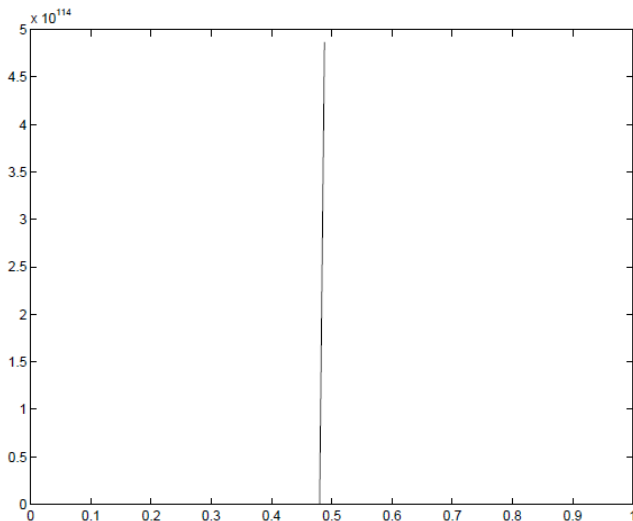


Figure 4: Lax-Wendroff method for Two-Variable Nonlinear Equation (a)  $\alpha = :8$ ; (b)  $\alpha = :1$ :

previous methods, the solution is stable and has a very small magnitude of error for a Courant factor of .1.

### Leapfrog Method

How It Works Unfortunately, we have a second derivative in this system. Thus we must evaluate both  $g$  and  $K$  on whole steps, instead of using half steps as we have in previous methods. Thus our difference equations are

$$g_i^{n+1} = g_i^{n-1} + 2\tau K_i^n \text{ and}$$

$$K_i^{n+1} = K_i^{n-1} + 2\tau \left( \frac{g_{i+1}^n - 2g_i^n + g_{i-1}^n}{h^2} - \frac{(K_i^n)^2}{g_i^n} \right).$$

Solutions Again, we see that the three-variable system is stable for higher courant factors but that the two-variable system has a smaller relative error for  $\alpha = :1$ :

In Figure 5(a), the solution blows up before .6 crossing times as opposed to the 50 crossing times it took in Figure 2(a). The important thing is, however, that Leapfrog is stable (at least to 100 crossing times) for a small Courant factor (Figure 5(b)).

## Iterative Crank-Nicholson Method

How It Works Again recall that for the Iterated Crank-Nicholson method, we took each term in the second term on the right hand side of the FTCS method and replaced it by the average of that value and the value of the function at the next

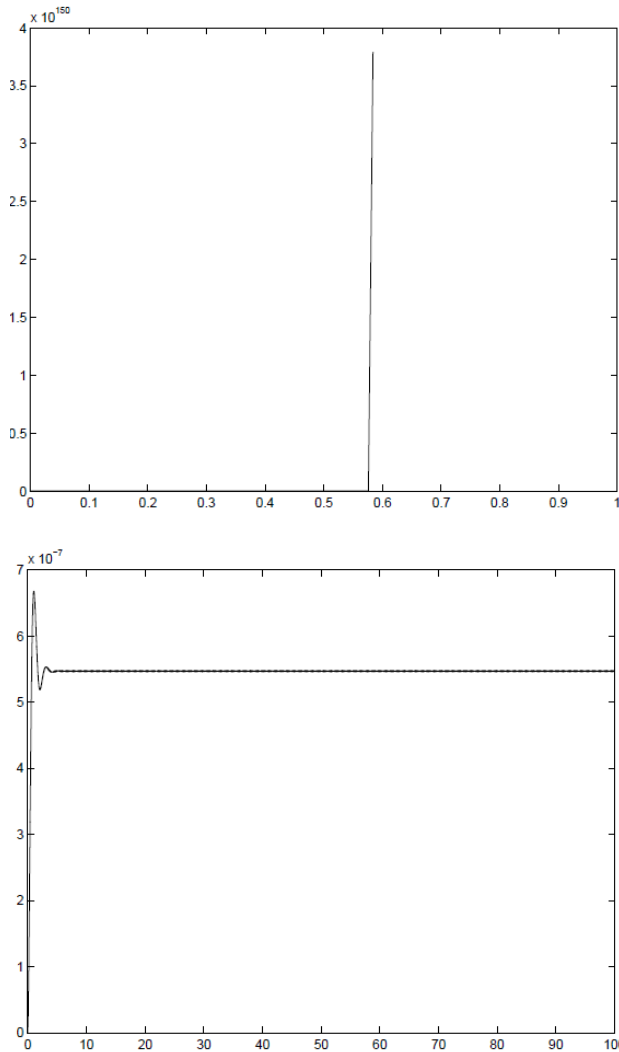


Figure 5: Leapfrog method for Two-Variable Nonlinear Equation (a)  $\alpha = .8$ ; (b)  $\alpha = .1$

time step. Thus our difference equations are

$$g_i^{n+1} = g_i^n + \frac{\tau}{2} (K_i^{n+1} + K_i^n) \text{ and}$$

$$g_i^{n+1} = g_i^n + \frac{\tau}{2} \left( \frac{g_{i+1}^{n+1} - 2g_i^{n+1} + g_{i-1}^{n+1}}{h^2} - \frac{(K_i^{n+1})^2}{g_i^{n+1}} + \frac{g_{i+1}^n - 2g_i^n + g_{i-1}^n}{h^2} - \frac{(K_i^n)^2}{g_i^n} \right)$$

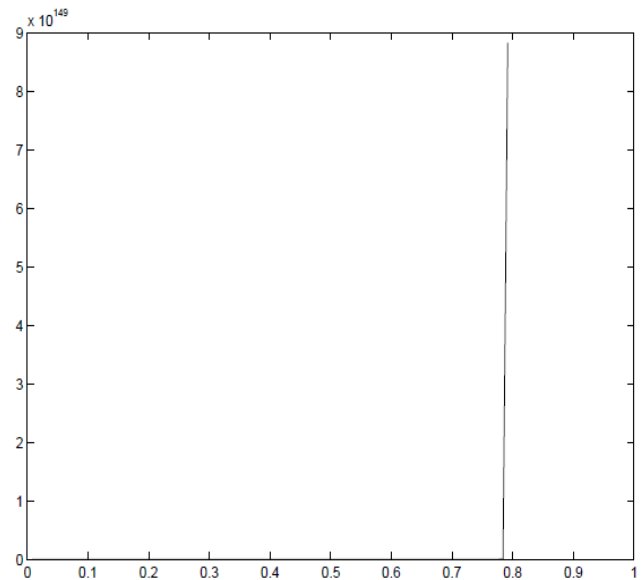
**Solutions** Similar to the previous results, the three-variable system is better for a Courant factor of .8. However, this time the two-variable system is worse for a Courant factor of .1. Consider Figure 6(b), though.

Notice the magnitude of error. The error is on the order of  $10^{-2}$ , which is much larger than the  $10^{-6}$  errors we were seeing previously. Thus, Crank-Nicholson is not such a great approach for the two-variable system.

### Summary

In all of the methods, we saw that the three-variable system was a better approximation than the two-variable system for a Courant factor of .8 and therefore is more stable at higher Courant factors. However, the two-variable system was a better approximation at a Courant factor of .1 for the Lax-Wendroff and Leapfrog methods.

This most likely has to do with the second-derivative in the latter system. However, for all practical purposes, all of the methods above would work, provided we used a small Courant factor.



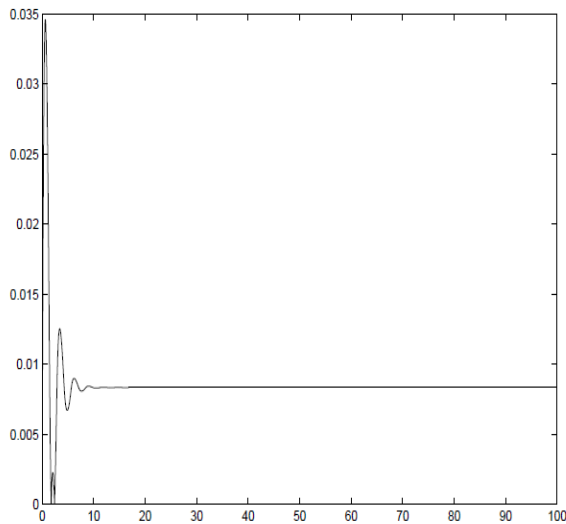


Figure 6: Crank-Nicholson method for Two-Variable Nonlinear Equation (a)  $\alpha = :8$ ; (b)  $\alpha = :1$

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