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Coincidence and Common Fixed Point Using Sequentially Weak Compatible Mapping

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Abstract.

In this paper we are going to introduce the concept of sequentially weak contraction using sequence of function which is uniformly convergent to a continuous function . The concept of sequence of function is already given by Dutta et. al.[5].

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1. Introduction

Stability of fixed points of contraction mappings has been studied by Bonsall et.al. [2] and Nadler et.al. [11]. These authors consider a sequence (T_n) of maps defined on a metric space (X,d) into itself and study the convergence of the sequence of fixed points for uniform or pointwise convergence of (T_n), under contraction assumptions of the maps.

Theorem.1.1.

Let (X, d) be a complete metric space and let T : X → X be a self-mapping satisfying the inequality

(1.1) $\psi(d(Tx, Ty)) \le \psi(d(x, y)) - \varphi(d(x, y))$

where $\varphi, \psi : [0, \infty) \to [0, \infty)$ are both continuous and monotonic nondecreasing functions with $\psi(t) = 0 = \varphi(t)$ if and only if t = 0.

Then T has a unique fixed point.

This theorem can be restated using sequence of function as:

Definition.1.2.

A mapping T : X → X, where (X, d) is a metric space, is said to be sequencially weakly contraction if

(1.2) $d(Tx, Ty) \le d(x, y) - f_n(d(x,y))$

$(f_n: I \text{ (interval or subset of R)} \to R)$

where $x,y \in X$ and $f_n(t)$ is a sequence of function which converges uniformly to t, and monotonic function such that $f_n(t) = 0$ if and only if t = 0.

If one takes $f_n(t) = kt$ where 0 < k < 1 and t = 1, then (1.2) reduces Banach Contraction Principle, which states that "Let (X, d) be a complete metric space. If T satisfies

(1.3) $d(Tx, Ty) \le k\, d(x, y)$

for each x, y in X, where $0 < k < 1$, then T has a unique fixed point in X."

Theorem 1.3.

Let (X, d) be a complete metric space and let : X → X be a self-mapping satisfying the inequality

(1.4) $\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - f_n(d(x, y))$

(f_n:I (interval or subset of R) → R) where $f_n(t)$ is a monotonically non decreasing sequence of function which converges uniformly to $\psi(t)$, then T has a unique fixed point. Where $\psi : [0,\infty) \to [0,\infty)$ is continuous and monotonically non-decreasing and continuous function. Then T has a unique fixed point.

Proof: For any $x_0 \in X$, we construct a sequence $\{x_n\}$ by,

$$x_n = Tx_{n-1}\ , n = 1,2,3,4\ldots\ldots.$$

substituting $x = x_{n-1}$ and $y = x_n$ in (1.4), we obtain

(1.5) $\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - f_n(d(x_{n-1}, x_n))$

Which implies,

(1.6) $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$

(using monotonic property of ψ -function) it follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotonically decreasing and consequently there exist $r \geq 0$ such that

(1.7) $d(x_n, x_{n+1}) \to r$ as $n \to \infty$

letting $n \to \infty$ in (1.5), we obtain,

(1.8) $\psi(r) \leq \psi(r) - \psi(r)$

since, $\lim_{n\to\infty} f_n(r) = \psi(r)$

Which is a contradiction unless r = 0, since $\psi(r) \geq 0$ Hence

(1.9) $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$

we next prove that $\{x_n\}$ is a Cauchy sequence.

If possible let $\{x_n\}$ is not Cauchy sequence then there exist $\varepsilon > 0$ for which we can find subsequence $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

(1.10) $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$

further corresponding to m(k), we can choose n(k) in such a way that it is a smallest integer with $n(k) > m(k)$ and satisfying (1.10) then

(1.11) $d(x_{m(k)}, x_{n(k)-1}) < \varepsilon$

then we have,

(1.12) $\varepsilon \leq d(x_{m(k)}, x_{n(k)})$
$\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})$
$< \varepsilon + d(x_{n(k)-1}, x_{n(k)})$

letting $k \to \infty$ and using (1.9), we have

(1.13) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) < \varepsilon$

again,

(1.14) $d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + (x_{m(k)}, x_{m(k)-1})$

letting $k \to \infty$ in the above inequalities and using (1.9), (1.13) we get

(1.15) $\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon$

setting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (1.4) and using (1.10) we obtain,

(1.16) $\psi(\varepsilon) \leq \psi(d(x_{m(k)}, x_{n(k)}))$
$\leq \psi(d(x_{m(k)-1}, x_{n(k)-1})) - f_n(d(x_{m(k)-1}, x_{n(k)-1}))$

letting $k \to \infty$ in the above inequalities and using (1.13) and (1.15), we obtain

(1.17) $\psi(\varepsilon) \leq \psi(\varepsilon) - f_n(\varepsilon)$

Which is a contradiction if $\varepsilon > 0$.

Since $f_n(t)$ converges uniformly to $\psi(\varepsilon)$.

This shows that $\{x_n\}$ is a Cauchy sequence and hence is convergent in the complete metric space X.

(1.18) Let $x_n \to z$ (say) as $n \to \infty$

Substituting $x = x_{n-1}$ and $y = z$ in (1.4), we obtain

(1.19) $\psi(d(x_n,Tz)) \leq \psi(d(x_{n-1},z)) - f_n(d(x_{n-1},z)$

letting $n \to \infty$, using (1.18) and continuity of ψ and continuity of f_n at infinity we have

$$\psi(d(z,Tz)) \leq \psi(0) - \lim_{n\to\infty} \{f_n(0)\}$$
$$\leq \psi(0) - \psi(0)$$
$$= 0$$

(1.20) Which implies, $\psi(d(z, Tz)) = 0$
i.e $d(z,Tz) = 0$

(1.21) or $z = Tz$

To prove uniqueness of fixed point, let z_1 and z_2 are two fixed points of T

Putting $x = z_1$ and $y = z_2$ in (1.4),

$$\psi(d(Tz_1, Tz_2)) \leq \psi(d(z_1, z_2)) - f_n(d(z_1, z_2))$$
or
$$\psi(d(z_1, z_2)) \leq \psi(d(z_1, z_2)) - f_n(d(z_1, z_2))$$
[using (1.21)]

or $\psi(d(z_1, z_2)) \leq 0$

Since $f_n(t)$ converges uniformly to $\psi(\varepsilon)$.

or

equivalently $d(z_1, z_2) = 0$, i.e., $z_1 = z_2$.

This proves the uniqueness of fixed point.

In 2006, Beg et. al. [1] generalized Theorem (1.1) in the following form:

Theorem 1.4.

Let (X, d) be a metric space and let f be a weakly contractive mapping with respect to g, that is,

(1.22) $\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \varphi(d(gx,gy))$

for all $x, y \in X$.

Where $\varphi, \psi : [0, \infty) \to [0, \infty)$ are two mappings with $\varphi(0) = \psi(0) = 0$, ψ is continuous nondecreasing and φ is lower semi-continuous.

If $fX \subset gX$ and gX is a complete subspace of X, then f and g have coincidence point in X.

In 2012, Moradi et. al. [10] proved the following Theorems:

Theorem 1.5.

Let T be self mapping on a complete metric space (X, d) satisfying the following:

(1.23) $\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi_n(d(x, y))$,

for all $x, y \in X$ ($(\psi - \varphi)$ weakly contractive), where $\varphi_n(t)$ is a sequence of function which converges to $\psi(t)$. Also suppose that either

(i) ψ is continuous and $\lim_{n\to\infty} t_n = 0$, if $\lim_{n\to\infty} \varphi(t_n) = 0$.
or
(ii) ψ is monotonic non-decreasing and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} \varphi(t_n) = 0$.

Then T has a unique fixed point.

Now, we prove our results on metric space for pair of sequencially weak compatible mappings.

2. Main result:

Theorem 2.1.

Let f and g be self mappings on a metric space (X, d) satisfying the followings:

(2.1) $gX \subset fX$,

(2.2) gX or fX is complete,

(2.3) $\psi(d(gx, gy)) \leq \psi(d(fx, fy)) - f_n(d(fx,fy))$,

(f_n : I (interval or subset of R) → R) for all x, y ∈ X

where $\psi : [0, \infty) \to [0, \infty)$ is mappings with $\psi(0) = 0$, $f_n(t) > 0$ also, $f_n(t)$ is a uniformally convergent sequence which converges to $\psi(t)$ and $\psi(t) > 0$ for all t > 0.

Suppose also that either

(a) ψ is continuous and $\lim_{n\to\infty} t_n = 0$, if $\lim_{n\to\infty} f_n(t_n) = 0$.
or
(b) ψ is monotonic non-decreasing and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} f_n(t_n) = 0$.

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. From (2.1), one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X by

$y_n = fx_{n+1} = gx_n$, n = 0, 1, 2, ...

Moreover, we assume that if $y_n = y_{n+1}$ for some n ∈ ℕ, then there is nothing to prove. Now, we assume that $y_n \neq y_{n+1}$ for all n ∈ ℕ.

Substituting $x = x_{n+1}$ and $y = x_n$ in (2.3), we have

(2.4) $$\begin{aligned}\psi(d(y_{n+1}, y_n)) &= \psi(d(gx_{n+1},gx_n))\\ &\leq \psi(d(fx_{n+1}, fx_n)) - f_n(d(fx_{n+1},fx_n))\\ &= \psi(d(y_n, y_{n-1})) - f_n(d(y_n,y_{n-1}))\end{aligned}$$

for all n ∈ ℕ and hence the sequence $\{\psi(d(y_{n+1}, y_n))\}$ is monotonic decreasing and bounded below. Thus, there exists $r \geq 0$ such that $\lim_{n\to\infty}\psi(d(y_{n+1}, y_n)) = r$.

From (2.4), we deduce that

(2.5) $$0 \leq f_n(d(y_n,y_{n-1})) \leq \psi(d(y_n, y_{n-1})) - \psi(d(y_{n+1}, y_n)).$$

Letting n → ∞ in the above inequality,

we get $\lim_{n\to\infty} f_n((d(y_n, y_{n-1})) = 0$.

If (a) holds, then by hypothesis

$$\lim_{n\to\infty} d(y_n, y_{n-1}) = 0.$$

If (b) holds, then from (2.5), we have

$d(y_{n+1}, y_n) < d(y_n, y_{n-1})$, for all n∈ ℕ.

Hence $\{d(y_{n+1}, y_n)\}$ is monotonically decreasing and bounded below.

By hypothesis, $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$.

Therefore, in every case, we conclude that

(2.6) $$\lim_{n\to\infty} d(y_n, y_{n-1}) = 0.$$

Now, we claim that $\{y_n\}$ is a Cauchy sequence. Indeed, if it is false, then there exists $\varepsilon > 0$ and the subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that n(k) is minimal in the sense that n(k) > m(k) > k and $d(y_{m(k)}, y_{n(k)}) \geq \varepsilon$ and by using the triangular inequality, we obtain

$$\begin{aligned}\varepsilon &\leq d(y_{m(k)}, y_{n(k)})\\ &\leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})\\ &\leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})\\ &< 2d(y_{m(k)}, y_{m(k)-1}) + \varepsilon + d(y_{n(k)-1}, y_{n(k)}).\end{aligned}$$

2.7 $$\varepsilon < 2d(y_{m(k)}, y_{m(k)-1}) + \varepsilon + d(y_{n(k)-1}, y_{n(k)}).$$

Letting $k \to \infty$ in the above inequality and using (2.6), we get

(2.8) $\lim_{k\to\infty} (d(y_{m(k)},y_{n(k)}) = \lim_{k\to\infty} (d(y_{m(k)-1},y_{n(k)-1})$
$= \varepsilon.$

For all $k \in \mathbb{N}$, from (2.3), we have

(2.9) $\psi(d(y_{m(k)}, y_{n(k)})) \le \psi(d(y_{m(k)-1}, y_{n(k)-1})) - f_n(d(y_{m(k)-1},y_{n(k)-1}))$

If (a) holds, then

$\lim_{k\to\infty}\psi(d(y_{m(k)-1},y_{n(k)-1})) = \lim_{k\to\infty} \psi(d(y_{m(k)}, y_{n(k)}))$
$= \psi(\varepsilon),$

Now, from (2.9), we conclude that

$\lim_{k\to\infty} f_n(d(y_{m(k)-1}, y_{n(k)-1}) = 0.$

By hypothesis $\lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)-1}) = 0$, a contradiction. (Using (2.8))

If (b) holds, then from (2.9), we have

$\varepsilon < d(y_{m(k)}, y_{n(k)}) < d(y_{m(k)-1}, y_{n(k)-1}),$
and so

$d(y_{m(k)}, y_{n(k)}) \to \varepsilon^+$ and

$d(y_{m(k)-1}, y_{n(k)-1}) \to \varepsilon^+$ as $k \to \infty$.

Hence $\lim_{k\to\infty}\psi(d(y_{m(k)-1}, y_{n(k)-1}))$

$= \lim_{k\to\infty}\psi(d(y_{m(k)}, y_{n(k)})) = \psi(\varepsilon^+),$

where $\psi(\varepsilon^+)$ is the right limit of ψ at ε.

Therefore, from (2.9),

we get $\lim_{k\to\infty} f_n(d(y_{m(k)-1}, y_{n(k)-1})) = 0$.

By hypothesis $\lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)-1}) = 0$, a contradiction.

Thus $\{y_n\}$ is a Cauchy sequence.

Since fX is complete, so there exists a point $z \in fX$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{n+1} = z$.

Now, we show that z is the common fixed point of f and g. Since $z \in fX$, so there exists a point

$p \in X$ such that $fp = z$.

If (a) holds, then from (2.3), for all $n \in \mathbb{N}$, we have

$\psi(d(fp,gp)) = \lim_{n\to\infty}(d(gp,gx_n))$

$\le \lim_{n\to\infty}\psi(d(fp, fx_n))$

$- \lim_{k\to\infty} f_n (d(fp,fx_n))$

$\le \lim_{n\to\infty} \psi(d(fp, fx_n)).$

(2.10) $\psi(d(fp, gp)) \le \lim_{n\to\infty} \psi(d(fp, fx_n)).$

Using condition (a) and $\lim_{n\to\infty} y_n = z$, we get

$\psi(d(fp, gp)) \le \psi(d(z, z)) = \psi(0) = 0$ and so $d(gp, fp) = 0$ (note that f_n and ψ are non-negative with $f_n(0) = \psi(0) = 0$), which implies that $gp = fp = z$.

If (b) holds, then from (2.7), we have

$\psi(d(fp, gp)) = \lim_{n\to\infty}\psi(d(gp,gx_n))$

$\le \lim_{n\to\infty}\psi(d(fp, fx_n))$

$- \lim_{k\to\infty} f_n(d(fp,fx_n))$

(2.11) $\psi(d(fp, gp)) = 0$

(since f_n converges uniformly to ψ)

$d(fp,gp)=0$, which implies that $fp=gp=z$(say).

Now, we show that $z = fp = gp$ is a common fixed point of f and g. Since $fp = gp$ and f, g are weakly compatible maps, we have $fz = fgp = gfp = gz$.

We claim that $fz = gz = z$.

Let, if possible, $gz \ne z$.

If (a) holds, then from (2.3), we have

ψ(d(gz, z)) = ψ(d(gz, gp))

$\leq \psi$(d(fz, fp)) – f_n(d(fz,fp))

= ψ(d(gz, z)) – f_n(d(gz,z))

<ψ(d(gz, z)), a contradiction.

If (b) holds, then we have

d(gz, z) < d(gz, z), a contradiction.

Hence gz = z = fz, so z is the common fixed point of f and g.

For the uniqueness, let u be another common fixed point of f and g, so that fu = gu = u.

We claim that z = u.

Let, if possible, z $\neq$ u.

If (a) holds and $n \to \infty$ then from (2.3), we have

ψ(d(z, u)) = ψ(d(gz, gu))

$\leq \psi$(d(fz, fu)) – f_n(d(fz,fu))

= ψ(d(z, u)) –f_n(d(z,u))

< ψ(d(z, u)), a contradiction.

If (b) holds, then we have

d(z, u) < d(z, u), a contradiction.

Thus,(d(z, u)=0 i.e we get z = u.

Hence z is the unique common fixed point of f and g.

Example 2.2. Let X = [0, 1] be endowed with the Euclidean metric d(x, y) = $|x-y|$ for all x, y in X and let gx = (1/5)x and fx =(3/5)x for each x $\in$ X. Then

d(gx, gy) = 1/5 $|x-y|$ and

d(fx, fy) = $3/5|x-y|$.

Let ψ(t) = 5t and

f_n(t) = 25nt/(5n+t). Then

ψ(d(gx, gy)) = ψ(1/5$|x-y|$) = $|x-y|$

ψ(d(fx, fy)) = $\psi(\frac{3}{5}(|x-y|$

= 5(3/5$|x-y|$)

= 3$|x-y|$

f_n((d(fx, fy)) = 15n$|x-y|$ /(5n+$|x-y|$). also f_n(x) is a sequence of function which uniformly converges to $\psi(x)$.

Now

ψ(d(fx, fy))-f_n((d(fx, fy))

= 3$|x-y|$-15n$|x-y|$ /(5n+$|x-y|$)

= 3$|x-y|$[1-5n /(5n+$|x-y|$)]

And [1-5n /(5n+$|x-y|$)] $\geq$0 if n approaches to infinity.

So ψ(d(gx, gy)) < ψ(d(fx, fy))-f_n((d(fx, fy)).

From here, we conclude that f, g satisfy the relation (2.3).

Also gX = [0, $\frac{1}{5}$] $\subseteq$ [0, $\frac{3}{5}$] = fX, f and g are weakly compatible and 0 is the unique common fixed point of f and g.

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