International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 01
January 2015

Method for Solving the Differential Problem Related to Sine and Cosine Functions

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Abstract

In this paper, we study the differential problems related to sine and cosine functions. The analytic forms of any order derivatives of two types of functions can be determined by using binomial series and differentiation term by term theorem. Moreover, two examples are proposed to demonstrate the calculations. The research method adopted in this study is to find the solutions through manual calculations and verify these answers using Maple.

Key Words: derivatives; sine and cosine functions, analytic forms; binomial series; differentiation term by term theorem; Maple

1. Introduction

In calculus and engineering mathematics courses, finding $f^{(n)}(c)$ (the n -th order derivative value of function $f(x)$ at $x=c$), in general, necessary goes through two procedures: Evaluating $f^{(n)}(x)$ (the n-th order derivative of $f(x)$), and substituting $x=c$ to $f^{(n)}(x)$. When evaluating the higher order derivative values of a function (i.e. n is large), these two procedures will become more complicated. Therefore, to obtain the answers through manual calculations is very difficult. Griewank and Walther [1] introduced the computation of derivatives using automatic differentiation. Yu [2-10], Yu and B. -H. Chen [11], and Yu and T. -J. Chen [12] used many methods (for example, geometric series, binomial theorem, binomial series, and differentiation term by term theorem) to find the derivatives of some functions. In this paper, we evaluate the derivatives of the following two types of functions related to sine and cosine functions,

$$f(x)=\cos^{m}(cx+d)\cdot[a+b\cos^{n}(cx+d)]^{r}, \tag{1}$$

$$g(x)=\sin^{m}(cx+d)\cdot[a+b\sin^{n}(cx+d)]^{r}, \tag{2}$$

where a,b,c,d,r are real numbers, $a>|b|$, and m,n are non-negative integers. We can obtain the analytic forms of any order derivatives of these two types of functions by using binomial series and differentiation term by term theorem; these are the major results of this paper (i.e., Theorems 1 and 2), and hence greatly reduce the difficulty of calculating their higher order derivative values. On the other hand, we propose some examples to do calculation practically. The research methods adopted in this study is to find the solutions through manual calculations and verify these solutions using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights.

2. Main Results

First, some notations, formulas and theorems used in this paper are introduced below.

2.1 Notations:
Suppose that r is a real number, define $(r)_k = r(r-1)\cdots(r-k+1)$ for any positive integer $k \le r$, and $(r)_0 = 1$.

2.2 Formulas:

2.2.1 Euler's formula :
$e^{i\theta} = \cos\theta + i\sin\theta$, where θ is any real number.
2.2.2 DeMoivre's formula **:**
$(\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta$,
where p is any integer, and θ is any real number.

2.3 Theorems:
The followings are two important theorems used in this study.

2.3.1 Binomial series ([13, p 244]):
$(1+y)^r = \sum_{k=0}^{\infty} \frac{(r)_k}{k!} y^k$, where r, y are real numbers, and $|y| < 1$.

2.3.2 Differentiation term by term theorem ([13, p 230]):
If, for all non-negative integer k, the functions $g_k : (a,b) \to R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b), (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b). Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b). Moreover, its derivative $\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

In the following, we determine the analytic forms of any order derivatives of the function (1).

Theorem 1 *Suppose that* a,b,c,d,r *are real numbers,* $a > |b|$, m,n,q *are non-negative integers, and let the domain of* $f(x) = \cos^m(cx+d)\cdot[a + b\cos^n(cx+d)]^r$ *be* R, *then the* q*-th order derivative of* $f(x)$,

$$f^{(q)}(x) = a^r c^q \sum_{k=0}^{\infty} \sum_{p=0}^{nk+m} \frac{(r)_k (nk+m)_p}{k!\, p! \cdot 2^{nk+m}} \left(\frac{b}{a}\right)^k \times (nk+m-2p)^q \cos\left[(nk+m-2p)(cx+d) + \frac{q\pi}{2}\right], \tag{3}$$

for all $x \in R$.

Proof $f(x)$

$$= a^r \cos^m(cx+d)\left[1 + \frac{b}{a}\cos^n(cx+d)\right]^r$$

$$= a^r \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k \cos^{nk+m}(cx+d)$$

(by binomial series)

$$= a^r \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k \left[\frac{1}{2}[e^{i(cx+d)} + e^{-i(cx+d)}]\right]^{nk+m}$$ (by Euler's formula)

$$= a^r \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k \frac{1}{2^{nk+m}} \sum_{p=0}^{nk+m} \frac{(nk+m)_p}{p!} \times [e^{i(cx+d)}]^{nk+m-p}[e^{-i(cx+d)}]^p$$

$$= a^r \sum_{k=0}^{\infty} \frac{(r)_k}{k! \cdot 2^{nk+m}}\left(\frac{b}{a}\right)^k \sum_{p=0}^{nk+m} \frac{(nk+m)_p}{p!} \times e^{i(nk+m-2p)(cx+d)}$$

(by DeMoivre's formula)

$$= a^r \sum_{k=0}^{\infty} \sum_{p=0}^{nk+m} \frac{(r)_k (nk+m)_p}{k!\,p!\cdot 2^{nk+m}} \left(\frac{b}{a}\right)^k \times$$
$$\cos[(nk+m-2p)(cx+d)]. \tag{4}$$

Using differentiation term by term theorem, differentiating q times with respect to x on both sides of Eq. (4), we obtain Eq.(3).

q.e.d.

Next, the analytic forms of any order derivatives of the function (2) can be easily obtained by Theorem 1.

Theorem 2 *If the assumptions are the same as Theorem 1 and let the domain of* $g(x) = \sin^m(cx+d)\cdot[a+b\sin^n(cx+d)]^r$ *be* R*, then*

$$g^{(q)}(x) = a^r c^q \sum_{k=0}^{\infty} \sum_{p=0}^{nk+m} \frac{(r)_k (nk+m)_p}{k!\,p!\cdot 2^{nk+m}} \left(\frac{b}{a}\right)^k \times \tag{5}$$
$$(nk+m-2p)^q \cos\left[(nk+m-2p)\left(cx+d-\frac{\pi}{2}\right)+\frac{q\pi}{2}\right],$$

for all $x \in R$.

Proof Since $g(x) = \cos^m\left(cx+d-\frac{\pi}{2}\right)\times \left[a+b\cos^n\left(cx+d-\frac{\pi}{2}\right)\right]^r$, it follows from Eq. (4) that Eq. (5) holds. q.e.d.

3. Examples

In the following, for the differential problems discussed in this paper, two examples are proposed and we use Theorems 1 and 2 to obtain the analytic forms of their derivatives. Moreover, Maple is used to calculate the approximations of some of their higher order derivatives values to verify our answers.

Example 3.1 Suppose that the domain of $f(x) = \cos^3\left(2x+\frac{\pi}{4}\right)\left[7+4\cos^5\left(2x+\frac{\pi}{4}\right)\right]^6$ is R.

By Theorem 1, we obtain the q-th order derivative of $f(x)$,

$$f^{(q)}(x) = 7^6 2^q \sum_{k=0}^{\infty} \sum_{p=0}^{5k+3} \frac{(6)_k (5k+3)_p}{k!\,p!\cdot 2^{5k+3}} \left(\frac{4}{7}\right)^k \times$$
$$(5k+3-2p)^q \cos\left[(5k+3-2p)\left(2x+\frac{\pi}{4}\right)+\frac{q\pi}{2}\right], \tag{6}$$

for all $x \in R$.

Thus, the 4-th order derivative value of $f(x)$ at $x = \pi$,

$$f^{(4)}(\pi) = 7^6 2^4 \sum_{k=0}^{\infty} \sum_{p=0}^{5k+3} \frac{(6)_k (5k+3)_p}{k!\,p!\cdot 2^{5k+3}} \left(\frac{4}{7}\right)^k \times \tag{7}$$
$$(5k+3-2p)^4 \cos\frac{(5k+3-2p)\pi}{4}.$$

Next, we use Maple to verify the correctness of Eq. (7).

```
>f:=x->(cos(2*x+Pi/4))^3*(7+4*(cos(2*x+Pi/4))^5)^6;
```

```
>evalf((D@@4)(f)(Pi),14);
```

$$1.2747380355823\cdot 10^9$$

```
>evalf(7^6*2^4*(sum(product(3-s,s=0..(p-1))/(p!*8)*(3-2*p)^4*cos((3-2*p)*Pi/4),p=0..(5*k+3))+sum(sum(product(6-j,j=0..(k-1))*product(5*k+3-t,t=0..(p-1))/(k!*p!*2^(5*k+3))*(4/7)^k*(5*k+3-2*p)^4*cos((5*k+3-2*p)*Pi/4),p=0..(5*k+3)),k=1..infinity)),14);
```

$$1.2747380355820\cdot 10^9$$

Example 3.2 If the domain of $g(x)=\sin^5\left(3x-\frac{\pi}{6}\right)\left[8+3\sin^7\left(3x-\frac{\pi}{6}\right)\right]^9$ is R, then using Theorem 2 yields

$$g^{(q)}(x)=8^9 3^q\sum_{k=0}^{\infty}\sum_{p=0}^{7k+5}\frac{(9)_k(7k+5)_p}{k!\,p!\cdot 2^{7k+5}}\left(\frac{3}{8}\right)^k\times (7k+5-2p)^q\cos\left[(7k+5-2p)\left(3x-\frac{2\pi}{3}\right)+\frac{q\pi}{2}\right], \quad (8)$$

for all $x\in R$.

Therefore, the 4-th order derivative value of $g(x)$ at $x=\frac{\pi}{3}$,

$$g^{(4)}\left(\frac{\pi}{3}\right)=8^9 3^4\sum_{k=0}^{\infty}\sum_{p=0}^{7k+5}\frac{(9)_k(7k+5)_p}{k!\,p!\cdot 2^{7k+5}}\left(\frac{3}{8}\right)^k\times (7k+5-2p)^4\cos\frac{(7k+5-2p)\pi}{3}. \quad (9)$$

We also use Maple to verify the correctness of Eq. (9).

>g:=x->(sin(3*x-Pi/6))^5*(8+3*(sin(3*x-Pi /6))^7)^9;

>evalf((D@@4)(g)(Pi/3),14);

$$7.5367277293953\cdot 10^{11}$$

>evalf(8^9*3^4*(sum(product(5-s,s=0..(p-1))/(p!*32)*(5-2*p)^4*cos((5-2*p)*Pi/3),p=0..(7*k+5))+sum(sum(product(9-j,j=0..(k-1))*product(7*k+5-t,t=0..(p-1))/(k!*p!*2^(7*k+5))*(3/8)^k*(7*k+5-2*p)^4*cos((7*k+5-2*p)*Pi/3),p=0..(7*k+5)),k=1..infinity)),14);

$$7.5367277293954\cdot 10^{11}$$

4. Conclusion

In this paper, we provide some techniques to evaluate the derivatives of functions related to sine and cosine functions. We hope these techniques can be applied to solve another differential problems. On the other hand, binomial series and differentiation term by term theorem play significant roles in the theoretical inferences of this study. In fact, the applications of the two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on further applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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