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Techniques for Evaluating Some Type of Multiple Improper Integral

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Abstract

This paper considers some type of multiple improper integral. We can obtain the closed form of this multiple improper integral using differentiation with respect to a parameter and Leibniz rule. On the other hand, some examples are proposed to demonstrate the calculations. The method adopted in this study is to find solutions through manual calculations and verify our answers using Maple. This method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking.

Key Words:

multiple improper integral; closed form; differentiation with respect to a parameter; Leibniz rule; Maple

1. Introduction

The multiple improper integral problem is closely related with probability theory and quantum field theory, research in this regard can refer to Streit [1] and Ryder [2]. For this reason, the evaluation and numerical calculation of multiple improper integrals are important, and can be studied based on Yu [3-8]. In this paper, we study the following multiple improper integral

$$\int_1^{\infty}\cdots\int_1^{\infty}[\ln(x_1+\cdots+x_n)]^k(x_1+\cdots+x_n)^a dx_1\cdots dx_n, \quad (1)$$

where a is a real number, n,k are positive integers, and $a<-n$. The closed form of this multiple improper integral can be determined by using differentiation with respect to a parameter and Leibniz rule; this is the major result of this study (i.e., Theorem A). In addition, two examples are used to demonstrate the proposed calculations. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods. For the instructions and operations of Maple can refer to [9-15].

2. Main Result

First, we introduce two important theorems used in this study which can be found in [16, p283]) and [16, p121] respectively.

2.1 Differentiation with respect to a parameter: Suppose that the $(n+1)$ variables function $f(\lambda,x_1,x_2,\cdots,x_n)$ is defined on $[\lambda_1,\lambda_2]\times I$. If $f(\lambda,x_1,x_2,\cdots,x_n)$ and its partial derivative $\frac{\partial f}{\partial\lambda}(\lambda,x_1,x_2,\cdots,x_n)$ are

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continuous functions on $[\lambda_1,\lambda_2]\times I$, and $\int_I \frac{\partial f}{\partial \lambda}(\lambda,x_1,\cdots,x_n)dx_1\cdots dx_n$ is uniformly convergent on the open interval (λ_1,λ_2). Then $F(\lambda)=\int_I f(\lambda,x_1,\cdots,x_n)dx_1\cdots dx_n$ is differentiable on (λ_1,λ_2). Moreover, $\frac{d}{d\lambda}F(\lambda)=\int_I \frac{\partial f}{\partial \lambda}(\lambda,x_1,\cdots,x_n)dx_1\cdots dx_n$ for $\lambda\in(\lambda_1,\lambda_2)$.

2.2 Leibniz rule: If m is a positive integer and $f(x),g(x)$ are functions such that their p-th derivatives $f^{(p)}(x),g^{(p)}(x)$ exist for all $p=1,\ldots,m$, then the formula of the m-th derivative of product function $f(x)g(x)$ is

$$(fg)^{(m)}(x)=\sum_{k=0}^{m}\frac{(m)_k}{k!}f^{(m-k)}(x)g^{(k)}(x),$$

where $(m)_k=m(m-1)(m-2)\cdots(m-k+1)$ for $k=1,\cdots,m$, and $(m)_0=1$.

Before deriving the major result in this paper, the following two lemmas are needed.

Lemma 1 *If a is a real number, n is a positive integer, and $a\neq -1,-2,\cdots,-n$, then*

$$\frac{1}{(a+1)(a+2)\cdots(a+n)}=\sum_{p=1}^{n}\frac{(-1)^{p-1}}{(n-p)!(p-1)!(a+p)}. \quad (2)$$

Proof Let

$$\frac{1}{(a+1)(a+2)\cdots(a+n)}=\sum_{p=1}^{n}\frac{A_p}{(a+p)},$$

where A_p is a constant for all $p=1,\cdots,n$.

It follows that

$$\sum_{p=1}^{n}A_p\frac{(a+1)(a+2)\cdots(a+n)}{(a+p)}=1.$$

Hence, $A_p=\frac{(-1)^{p-1}}{(n-p)!(p-1)!}$. Therefore, Eq. (2) holds. q.e.d.

Lemma 2 *If a is a real number, n is a positive integer, and $a<-n$, then*

$$\int_1^\infty\cdots\int_1^\infty(x_1+\cdots+x_n)^a dx_1\cdots dx_n$$
$$=(-1)^n\sum_{p=1}^{n}\frac{(-1)^{p-1}}{(n-p)!(p-1)!(a+p)}\cdot n^{a+n}. \quad (3)$$

Proof $\int_1^\infty\cdots\int_1^\infty(x_1+\cdots+x_n)^a dx_1\cdots dx_n$

$$=\int_1^\infty\cdots\int_1^\infty\left(\frac{1}{a+1}(x_1+x_2+\cdots+x_n)^{a+1}\Big|_{x_1=1}^{x_1=\infty}\right)dx_2\cdots dx_n$$
$$=\frac{-1}{a+1}\int_1^\infty\cdots\int_1^\infty(1+x_2+\cdots+x_n)^{a+1}dx_2\cdots dx_n$$
$$=\frac{-1}{a+1}\int_1^\infty\cdots\int_1^\infty\left(\frac{1}{a+2}(1+x_2+x_3+\cdots+x_n)^{a+2}\Big|_{x_2=1}^{x_2=\infty}\right)dx_3\cdots dx_n$$
$$=\frac{1}{(a+1)(a+2)}\int_1^\infty\cdots\int_1^\infty(2+x_3+\cdots+x_n)^{a+2}dx_3\cdots dx_n$$
$$=\frac{(-1)^{n-1}}{(a+1)(a+2)\cdots(a+n-1)}\int_1^\infty(n-1+x_n)^{a+n-1}dx_n$$
$$=\frac{(-1)^{n-1}}{(a+1)(a+2)\cdots(a+n-1)}\left(\frac{1}{a+n}(n-1+x_n)^{a+n}\Big|_{x_n=1}^{x_n=\infty}\right)$$
$$=(-1)^n\frac{1}{(a+1)(a+2)\cdots(a+n)}\cdot n^{a+n}$$
$$=(-1)^n\sum_{p=1}^{n}\frac{(-1)^{p-1}}{(n-p)!(p-1)!(a+p)}\cdot n^{a+n}.$$

(by Lemma 1) q.e.d.

In the following, we determine the closed form of the multiple improper integral (1).

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Theorem A *Suppose that* a *is a real number,* n, k *are positive integers, and* $a < -n$ *, then*

$$\int_1^\infty \cdots \int_1^\infty [\ln(x_1 + \cdots + x_n)]^k (x_1 + \cdots + x_n)^a dx_1 \cdots dx_n$$

$$= (-1)^n n^{a+n} \sum_{p=1}^{n} \sum_{q=0}^{k} \frac{(-1)^{p+k-q-1}(k)_q (k-q)!}{(n-p)!(p-1)!q!(a+p)^{k-q+1}} (\ln n)^q. \tag{4}$$

Proof Using differentiation with respect to a parameter and Leibniz rule, differentiating k times with respect to a on both sides of Eq. (3) yields

$$\int_1^\infty \cdots \int_1^\infty [\ln(x_1 + \cdots + x_n)]^k (x_1 + \cdots + x_n)^a dx_1 \cdots dx_n$$

$$= (-1)^n \sum_{p=1}^{n} \frac{(-1)^{p-1}}{(n-p)!(p-1)!} \sum_{q=0}^{k} \frac{(k)_q}{q!} \left(\frac{1}{a+p} \right)^{(k-q)} (n^{a+n})^{(q)}$$

$$= (-1)^n \sum_{p=1}^{n} \frac{(-1)^{p-1}}{(n-p)!(p-1)!} \sum_{q=0}^{k} \frac{(k)_q}{q!} \frac{(-1)^{k-q}(k-q)!}{(a+p)^{k-q+1}} (\ln n)^q n^{a+n}$$

$$= (-1)^n n^{a+n} \sum_{p=1}^{n} \sum_{q=0}^{k} \frac{(-1)^{p+k-q-1}(k)_q (k-q)!}{(n-p)!(p-1)!q!(a+p)^{k-q+1}} (\ln n)^q.$$

q.e.d.

3. Example

In the following, for the multiple improper integral problem discussed in this study, we propose two examples and use Theorem A to obtain their closed forms. Additionally, Maple is used to calculate the approximations of these multiple improper integrals and their solutions to verify our answers.

Example 1 By Eq. (4), we have the following double improper integral

$$\int_1^\infty \int_1^\infty [\ln(x_1 + x_2)]^4 (x_1 + x_2)^{-3} dx_1 dx_2$$

$$= \frac{1}{2} \sum_{p=1}^{2} \sum_{q=0}^{4} \frac{(-1)^{p-q+3}(4)_q (4-q)!}{(2-p)!(p-1)!q!(-3+p)^{5-q}} (\ln 2)^q. \tag{5}$$

Next, we use Maple to verify the correctness of Eq. (5).

>evalf(Doubleint((ln(x1+x2))^4*(x1+x2)^(-3),x1=1..infinity,x2=1..infinity),14);

22.50252985678

>evalf(1/2*sum(sum((-1)^(p-q+3)*product(4-j,j=0..(q-1))*(4-q)!/((2-p)!*(p-1)!*q!*(-3+p)^(5-q))*(ln(2))^q,q=0..4),p=1..2),14);

22.50252985699

Example 2 On the other hand, using Eq. (4) yields the triple improper integral

$$\int_1^\infty \int_1^\infty \int_1^\infty [\ln(x_1 + x_2 + x_3)]^3 (x_1 + x_2 + x_3)^{-5} dx_1 dx_2 dx_3$$

$$= -\frac{1}{9} \sum_{p=1}^{3} \sum_{q=0}^{3} \frac{(-1)^{p-q+2}(3)_q (3-q)!}{(3-p)!(p-1)!q!(-5+p)^{4-q}} (\ln 3)^q. \tag{6}$$

We also use Maple to verify the correctness of Eq. (6).

>evalf(Tripleint((ln(x1+x2+x3))^3*(x1+x2+x3)^(-5),x1=1..infinity,x2=1..infinity,x3=1..infinity),14);

0.0625750425379

>evalf(-1/9*sum(sum((-1)^(p-q+2)*product(3-j,j=0..(q-1))*(3-q)!/((3-p)!*(p-1)!*q!*(-5+p)^(4-q))*(ln(3))^q,q=0..3),p=1..3),14);

0.0625750425380

4. Conclusion

This paper uses two techniques: differentiation with respect to a parameter and Leibniz rule to obtain the closed form of some type of multiple improper integral. In fact, the applications of the two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and employ Maple to solve these problems.

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