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Solving Real Integrals Using Complex Integrals

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Abstract

In this article, we study two types of real integrals of trigonometric functions. The closed forms of the two types of real integrals can be obtained using complex integrals. In addition, some examples are proposed to do a calculation practically. Simultaneously, Maple is used to calculate the approximations of some definite integrals and their solutions for verifying our answers.

Key Words: real integrals; trigonometric functions; closed forms; complex integrals; Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests.

In calculus and engineering mathematics, there are many methods to solve the integral problems, including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. This paper considers the following two types of integrals of trigonometric functions which are not easy to obtain their answers using the methods mentioned above.

$$\int \frac{\begin{bmatrix} -\alpha\beta r^2 \sin 2\theta \\ +[(\alpha+\beta-\omega)r^3+\alpha\beta\omega r]\sin\theta \end{bmatrix}}{(r^2-2\alpha r\cos\theta+\alpha^2)(r^2-2\beta r\cos\theta+\beta^2)}d\theta, \tag{1}$$

$$\int \frac{\begin{bmatrix} \alpha\beta r^2 \cos 2\theta \\ -[(\alpha+\beta+\omega)r^3+\alpha\beta\omega r]\cos\theta \\ +r^4+(\alpha+\beta)\omega r^2 \end{bmatrix}}{(r^2-2\alpha r\cos\theta+\alpha^2)(r^2-2\beta r\cos\theta+\beta^2)}d\theta, \tag{2}$$

where r,α,β,θ are real numbers, $\alpha \neq \beta$, and $r \neq \pm\alpha, \pm\beta$. The closed forms of the two types of integrals can be obtained using complex integral theory; these are the main results of this paper (i.e., Theorems 1 and 2). Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve the integral problems. On the other hand, Yu [4-27], Yu and Chen [28], and Yu and Sheu [29-31] used complex power series, integration term by term theorem, Parseval's theorem, area mean value theorem, and Cauchy integral formula to solve some types of integral problems. This paper uses some examples to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

2. Preliminaries and Results

Firstly, some definitions and complex integral formulas used in this paper are introduced below.

2.1 Definitions:

2.1.1 Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, and a, b are real numbers. a, the real part of z, is denoted as $\mathrm{Re}(z)$; b, the imaginary part of z, is denoted as $\mathrm{Im}(z)$.

2.1.2 The complex logarithmic function $\ln z$ is defined by $\ln z = \ln|z| + i\phi$, where z is a complex number, ϕ is a real number, $z = |z| \cdot e^{i\phi}$, and $-\pi < \phi \le \pi$.

2.2 Complex integral formula:

$\int \frac{1}{z-a} dz = \ln(z-a) + C$, where z, a are complex numbers, $z \ne a$ and C is a constant.

To obtain the major results, two lemmas are needed.

Lemma 1 *If α, β, ω are real numbers, $\alpha \ne \beta$, z is a complex number, and C is a constant, then the contour integral*

$$\int \frac{z-\omega}{(z-\alpha)(z-\beta)} dz$$

$$= \frac{\omega-\alpha}{\beta-\alpha} \ln(z-\alpha) + \frac{\beta-\omega}{\beta-\alpha} \ln(z-\beta) + C. \quad (3)$$

Proof $\int \frac{1}{(z-\alpha)(z-\beta)} dz$

$$= \int \left(\frac{\omega-\alpha}{\beta-\alpha} \cdot \frac{1}{z-\alpha} + \frac{\beta-\omega}{\beta-\alpha} \cdot \frac{1}{z-\beta} \right) dz$$

$$= \frac{\omega-\alpha}{\beta-\alpha} \ln(z-\alpha) + \frac{\beta-\omega}{\beta-\alpha} \ln(z-\beta) + C.$$

q.e.d.

Lemma 2 *Assume that r, θ, λ are real numbers and $r \ne \pm\lambda$, then*

$$\mathrm{Re}[\ln(re^{i\theta} - \lambda)] = \ln\sqrt{r^2 - 2\lambda r\cos\theta + \lambda^2}. \quad (4)$$

Moreover, if $r > 0$ and $0 < \theta < \pi$, then

$$\mathrm{Im}[\ln(re^{i\theta} - \lambda)] = \cot^{-1}\left(\frac{r\cos\theta - \lambda}{r\sin\theta} \right). \quad (5)$$

Proof Since $\ln(re^{i\theta} - \lambda)$

$$= \ln[(r\cos\theta - \lambda) + ir\sin\theta]$$

$$= \ln\left[\sqrt{(r\cos\theta-\lambda)^2 + r^2\sin^2\theta} \times \left(\frac{r\cos\theta-\lambda}{\sqrt{(r\cos\theta-\lambda)^2 + r^2\sin^2\theta}} + i\frac{r\sin\theta}{\sqrt{(r\cos\theta-\lambda)^2 + r^2\sin^2\theta}} \right) \right]$$

$$= \ln\sqrt{r^2 - 2\lambda r\cos\theta + \lambda^2}$$

$$+ \ln\left[\frac{r\cos\theta-\lambda}{\sqrt{r^2 - 2\lambda r\cos\theta + \lambda^2}} + i\frac{r\sin\theta}{\sqrt{r^2 - 2\lambda r\cos\theta + \lambda^2}} \right],$$

it follows from Definition 2.1.2 that the desired results hold. q.e.d.

Next, we obtain the closed form of the integral (1).

Theorem 1 *If r, α, β are real numbers, $\alpha \ne \beta$, and $r \ne \pm\alpha, \pm\beta$, then*

$$\int \frac{\begin{bmatrix} -\alpha\beta r^2 \sin 2\theta \\ +[(\alpha+\beta-\omega)r^3 + \alpha\beta\omega r]\sin\theta \end{bmatrix}}{(r^2 - 2\alpha r\cos\theta + \alpha^2)(r^2 - 2\beta r\cos\theta + \beta^2)} d\theta$$

$$= \frac{\omega-\alpha}{\beta-\alpha} \cdot \ln\sqrt{r^2 - 2\alpha r\cos\theta + \alpha^2} + \frac{\beta-\omega}{\beta-\alpha} \cdot \ln\sqrt{r^2 - 2\beta r\cos\theta + \beta^2} + C, \quad (6)$$

for all real numbers θ .

Proof Let $z = re^{i\theta}$ in Eq. (3), we have

$$\int \frac{(re^{i\theta}-\omega)ire^{i\theta}}{(re^{i\theta}-\alpha)(re^{i\theta}-\beta)}d\theta$$
$$= \frac{\omega-\alpha}{\beta-\alpha}\cdot \ln(re^{i\theta}-\alpha)$$
$$+\frac{\beta-\omega}{\beta-\alpha}\cdot \ln(re^{i\theta}-\beta)+C. \tag{7}$$

Using Eq. (4) and the equality of the real parts of both sides of Eq. (7) yields the desired result holds. q.e.d.

On the other hand, by Eq. (5) and the equality of the imaginary parts of both sides of Eq. (7), the closed form of the integral (2) can be easily obtained.

Theorem 2 *If the assumptions are the same as Theorem 1,* $r > 0$ *, and* $0 < \theta < \pi$ *, then*

$$\int \frac{\begin{bmatrix} \alpha\beta r^2\cos 2\theta \\ -[(\alpha+\beta+\omega)r^3+\alpha\beta\omega r]\cos\theta \\ +r^4+(\alpha+\beta)\omega r^2 \end{bmatrix}}{(r^2-2\alpha r\cos\theta+\alpha^2)(r^2-2\beta r\cos\theta+\beta^2)}d\theta$$
$$= \frac{\omega-\alpha}{\beta-\alpha}\cdot\cot^{-1}\left(\frac{r\cos\theta-\alpha}{r\sin\theta}\right)$$
$$+\frac{\beta-\omega}{\beta-\alpha}\cdot\cot^{-1}\left(\frac{r\cos\theta-\beta}{r\sin\theta}\right)+C. \tag{8}$$

3. Example

In the following, for the integral problems of trigonometric functions in this study, two examples are proposed and we use Theorems 1 and 2 to determine their closed forms. Moreover, we employ Maple to calculate the approximations of some definite integrals and their solutions for verifying our answers.

Example 3.1

Let $\alpha = 4, \beta = 2, \omega = 5, r = 3$ in Theorem 1, then using Eq. (6) Yields

$$\int \frac{-72\sin 2\theta + 147\sin\theta}{(25-24\cos\theta)(13-12\cos\theta)}d\theta$$
$$= -\frac{1}{2}\cdot \ln\sqrt{25-24\cos\theta}$$
$$+\frac{3}{2}\cdot\ln\sqrt{13-12\cos\theta}+C, \tag{9}$$

for all $\theta \in R$.
Hence, the definite integral

$$\int_{\pi/4}^{5\pi/3} \frac{-72\sin 2\theta + 147\sin\theta}{(25-24\cos\theta)(13-12\cos\theta)}d\theta$$
$$= -\frac{1}{4}\cdot\ln\left[\frac{25-24\cos(5\pi/3)}{25-24\cos(\pi/4)}\right]$$
$$+\frac{3}{4}\cdot\ln\left[\frac{13-12\cos(5\pi/3)}{13-12\cos(\pi/4)}\right]. \tag{10}$$

Next, we use Maple to verify the correctness of Eq. (10).

```
>evalf(int((-72*sin(2*theta)+147sin(theta))/
((25-24*cos(theta))*(13-12*cos(theta))),
theta=Pi/4..5*Pi/3),18);
```

0.208466734824125287

```
>evalf(-1/4*ln((25-24*cos(5*Pi/3))/(25-24
*cos(Pi/4)))+3/4*ln((13-12*cos(5*Pi/3))/(
13-12*cos(Pi/4))),18);
```

0.208466734824125286

Example 3.2

If $\alpha = 2, \beta = 6, \omega = 4, r = 1$ in Theorem 2, then by Eq. (8) we have

$$\int \frac{12\cos 2\theta - 60\cos\theta + 33}{(5-4\cos\theta)(37-12\cos\theta)}d\theta$$
$$= \frac{1}{2}\cdot\cot^{-1}\left(\frac{\cos\theta-2}{\sin\theta}\right)$$
$$+\frac{1}{2}\cdot\cot^{-1}\left(\frac{\cos\theta-6}{\sin\theta}\right)+C, \tag{11}$$

for $0 < \theta < \pi$.
Thus,

$$\int_{\pi/6}^{3\pi/4} \frac{12\cos 2\theta - 60\cos\theta + 33}{(5-4\cos\theta)(37-12\cos\theta)} d\theta$$

$$= \frac{1}{2} \cdot \left[\cot^{-1}\left(\frac{\cos(3\pi/4)-2}{\sin(3\pi/4)} \right) - \cot^{-1}\left(\frac{\cos(\pi/6)-2}{\sin(\pi/6)} \right) \right]$$

$$+ \frac{1}{2} \cdot \left[\cot^{-1}\left(\frac{\cos(3\pi/4)-6}{\sin(3\pi/4)} \right) - \cot^{-1}\left(\frac{\cos(\pi/6)-6}{\sin(\pi/6)} \right) \right]. \quad (12)$$

Using Maple to verify the correctness of Eq. (12) as follows:

```
>evalf(int((12*cos(2*theta)-60*cos(theta)+
33)/((5-4*cos(theta))*(37-12*cos(theta))),
theta=Pi/6..3*Pi/4),20);
```

0.075916818983840300065

```
>evalf(1/2*(arccot((cos(3*Pi/4)-2)/sin(3*Pi
/4))- arccot((cos(Pi/6)-2)/sin(Pi/6)))+ 1/2*(
arccot((cos(3*Pi/4)-6)/sin(3*Pi/4))- arccot((
cos(Pi/6)-6)/sin(Pi/6))),20);
```

0.075916818983840300071

4. Conclusion

In this study, we use complex integral theory to evaluate the real integrals of trigonometric functions. In fact, this technique can be applied to solve many integral problems. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topics to other calculus and engineering mathematics problems and solve these problems using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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