A Study of Partial Differential Problem with Maple

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Abstract

The present paper considers the partial differential problem of two types of two variables functions. The infinite series forms of any order partial derivatives of these two variables functions can be obtained mainly using binomial series and differentiation term by term theorem. Moreover, we propose some two variables functions to determine their partial derivatives and evaluate their higher order partial derivative values practically. At the same time, we employ Maple to calculate the approximations of these higher order partial derivative values and their infinite series forms for verifying our answers.

Key Words: two variables functions; partial derivatives; infinite series forms; binomial series; differentiation term by term theorem; Maple

1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Mozart, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This

study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics courses, the study of the partial differential problems of multivariable functions is important. For example, Laplace equation, wave equation, as well as other physical equations are involved the partial differentiation of multivariable functions. Therefore, in physics, engineering or other sciences, the evaluation and numerical calculation of the partial differentiation are valuable. Kalaba et.al [1], Neidinger [2], Bischof et al. [3], Fraenkel [4], and Griewank and Walther [5] provided some techniques to the computations of partial derivative values of multivariable functions. Rich [6], and Bischof et al. [7] used Matlab to the automatic differentiation. Yu [8-20], and Yu and Chen [21-22] used Maple and proposed some methods, for example, binomial series, Fourier series, complex power series, and differentiation term by term theorem to

evaluate arbitrary order partial derivative of some types of multivariable functions. In this paper, we study the partial differential problem of the following two types of two variables functions

$$f(x, y)$$

$$= \frac{\left[\begin{array}{l} \sum\limits_{s=0}^{m} \frac{(m)_s}{s!} (-\beta)^s \cos\left(\frac{s\pi}{2}\right)(\alpha x - \lambda)^{m-s} y^s \\ \times \sum\limits_{t=0}^{n} \frac{(n)_t}{t!} (-\beta)^t \cos\left(\frac{t\pi}{2}\right)(\alpha x - \mu)^{n-t} y^t \\ - \sum\limits_{s=0}^{m} \frac{(m)_s}{s!} (-\beta)^s \sin\left(\frac{s\pi}{2}\right)(\alpha x - \lambda)^{m-s} y^s \\ \times \sum\limits_{t=0}^{n} \frac{(n)_t}{t!} (-\beta)^t \sin\left(\frac{t\pi}{2}\right)(\alpha x - \mu)^{n-t} y^t \end{array}\right]}{[(\alpha x - \lambda)^2 + \beta^2 y^2]^m \cdot [(\alpha x - \mu)^2 + \beta^2 y^2]^n},$$

(1)

and

$$g(x, y)$$

$$= \frac{\left[\begin{array}{l} \sum\limits_{s=0}^{m} \frac{(m)_s}{s!} (-\beta)^s \cos\left(\frac{s\pi}{2}\right)(\alpha x - \lambda)^{m-s} y^s \\ \times \sum\limits_{t=0}^{n} \frac{(n)_t}{t!} (-\beta)^t \sin\left(\frac{t\pi}{2}\right)(\alpha x - \mu)^{n-t} y^t \\ + \sum\limits_{s=0}^{m} \frac{(m)_s}{s!} (-\beta)^s \sin\left(\frac{s\pi}{2}\right)(\alpha x - \lambda)^{m-s} y^s \\ \times \sum\limits_{t=0}^{n} \frac{(n)_t}{t!} (-\beta)^t \cos\left(\frac{t\pi}{2}\right)(\alpha x - \mu)^{n-t} y^t \end{array}\right]}{[(\alpha x - \lambda)^2 + \beta^2 y^2]^m \cdot [(\alpha x - \mu)^2 + \beta^2 y^2]^n},$$

(2)

where $\alpha, \beta, \lambda, \mu$ are real numbers, $\beta \neq 0$ and m, n are positive integers. The infinite series forms of any order partial derivatives of these two types of two variables functions can be obtained using binomial series and differentiation term by term theorem; these are the major results of this article (i.e., Theorems 1, 2). In addition, we propose two examples to demonstrate the calculations, and the answers are verified using Maple.

2. Preliminaries and Main Results

Some notations, formulas, and theorems used in this paper are introduced below.

2.1 Notations:

2.1.1 Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, and a, b are real numbers. a, the real part of z, is denoted as $\text{Re}(z)$; b, the imaginary part of z, is denoted as $\text{Im}(z)$.

2.1.2 Suppose that r is a real number, s is a positive integer, and $s \leq r$. Define $(r)_s = r(r-1)\cdots(r-s+1)$, and $(r)_0 = 1$.

2.1.3 Assume that p, q are non-negative integers. For the two-variables function $f(x, y)$, its q-times partial derivative with respect to x, and p-times partial derivative with respect to y, forms an $p + q$-th order partial derivative, and is denoted as $\frac{\partial^{p+q} f}{\partial y^p \partial x^q}(x, y)$.

2.2 Formulas and theorems:

2.2.1 Euler's formula:
$e^{i\theta} = \cos\theta + i\sin\theta$, where θ is a real number.

2.2.2 DeMoivre's formula:
$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$, where m is an integer, and θ is a real number.

2.2.3 Binomial series:
Suppose that u, v are complex numbers, a is a real number, and $|v| < |u|$, then

$$(u + v)^a = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} u^{a-k} v^k.$$

2.2.4 Differentiation term by term theorem ([23, p230]):

If, for all non-negative integer k, the functions $g_k : (a,b) \to R$ satisfy the following three conditions： (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b), (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b). Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b). Moreover, its derivative $\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) =$

$$\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x).$$

Before the major results can be derived, three lemmas are needed.

Lemma 1 *Suppose that a, b are real numbers, and r is a positive integer, then*

$$\operatorname{Re}[(a-ib)^r] = \sum_{s=0}^{r} \frac{(r)_s}{s!} \cos\left(\frac{s\pi}{2}\right) a^{r-s}(-b)^s,$$

(3)

$$\operatorname{Im}[(a-ib)^r] = \sum_{s=0}^{r} \frac{(r)_s}{s!} \sin\left(\frac{s\pi}{2}\right) a^{r-s}(-b)^s.$$

(4)

Proof $\operatorname{Re}[(a-ib)^r]$

$$= \operatorname{Re}\left[\sum_{s=0}^{r} \frac{(r)_s}{s!} a^{r-s}(-ib)^s \right]$$

$$= \operatorname{Re}\left[\sum_{s=0}^{r} \frac{(r)_s}{s!} \exp\left(\frac{is\pi}{2}\right) a^{r-s}(-b)^s \right]$$

(by Euler's formula and DeMoivre's formula)

$$= \sum_{s=0}^{r} \frac{(r)_s}{s!} \cos\left(\frac{s\pi}{2}\right) a^{r-s}(-b)^s.$$

Similarly,

$$\operatorname{Im}[(a-ib)^r]$$

$$= \operatorname{Im}\left[\sum_{s=0}^{r} \frac{(r)_s}{s!} \exp\left(\frac{is\pi}{2}\right) a^{r-s}(-b)^s \right]$$

$$= \sum_{s=0}^{r} \frac{(r)_s}{s!} \sin\left(\frac{s\pi}{2}\right) a^{r-s}(-b)^s. \qquad \text{q.e.d.}$$

Lemma 2 *If λ, μ are real numbers, z is a complex number, $z \neq \lambda, \mu$, $|z-\lambda| < |\lambda-\mu|$, and m, n are positive integers, then*

$$\frac{1}{(z-\lambda)^m (z-\mu)^n}$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} (\lambda-\mu)^{-n-k} (z-\lambda)^{k-m}. \quad (5)$$

Proof $\dfrac{1}{(z-\lambda)^m (z-\mu)^n}$

$$= \frac{[(z-\lambda)+(\lambda-\mu)]^{-n}}{(z-\lambda)^m}$$

$$= \frac{\sum_{k=0}^{\infty} \frac{(-n)_k}{k!} (\lambda-\mu)^{-n-k} (z-\lambda)^k}{(z-\lambda)^m}$$

(by binomial series)

$$= \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} (\lambda-\mu)^{-n-k} (z-\lambda)^{k-m}.$$

q.e.d.

Lemma 3 *Suppose that c, d are real numbers, $d > 0$, and u is an integer, then*

$$(c+id)^u = \left(\sqrt{c^2+d^2}\right)^u \exp\left(iu \cot^{-1} \frac{c}{d}\right).$$

(6)

International Journal of Research

Available at https://edupediapublications.org/journals

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 04
February 2016

Proof $(c+id)^u$

$$= \left(\sqrt{c^2+d^2}\right)^u \left(\frac{c}{\sqrt{c^2+d^2}}+i\frac{d}{\sqrt{c^2+d^2}}\right)^u$$

$$= \left(\sqrt{c^2+d^2}\right)^u (\cos\theta+i\sin\theta)^u$$

(where $\theta=\cot^{-1}\frac{c}{d}$)

$$= \left(\sqrt{c^2+d^2}\right)^u (\cos u\theta+i\sin u\theta)$$

(by DeMoivre's formula)

$$= \left(\sqrt{c^2+d^2}\right)^u e^{iu\theta}$$

(by Euler's formula)

$$= \left(\sqrt{c^2+d^2}\right)^u \exp\left(iu\cot^{-1}\frac{c}{d}\right).$$

q.e.d.

In the following, we determine the infinite series forms of any order partial derivatives of the two variables function (1).

Theorem 1 *Suppose that α,β,λ,μ are real numbers, $\beta\neq 0$, and m,n are positive integers. If the domain of the two variables function*

$$f(x,y)$$

$$= \frac{\left[\begin{array}{l}\sum\limits_{s=0}^{m}\frac{(m)_s}{s!}(-\beta)^s\cos\left(\frac{s\pi}{2}\right)(\alpha x-\lambda)^{m-s}y^s \\ \times\sum\limits_{t=0}^{n}\frac{(n)_t}{t!}(-\beta)^t\cos\left(\frac{t\pi}{2}\right)(\alpha x-\mu)^{n-t}y^t \\ -\sum\limits_{s=0}^{m}\frac{(m)_s}{s!}(-\beta)^s\sin\left(\frac{s\pi}{2}\right)(\alpha x-\lambda)^{m-s}y^s \\ \times\sum\limits_{t=0}^{n}\frac{(n)_t}{t!}(-\beta)^t\sin\left(\frac{t\pi}{2}\right)(\alpha x-\mu)^{n-t}y^t\end{array}\right]}{[(\alpha x-\lambda)^2+\beta^2 y^2]^m\cdot[(\alpha x-\mu)^2+\beta^2 y^2]^n} \ is$$

$$\left\{(x,y)\in R^2\Big|\beta y>0,\sqrt{(\alpha x-\lambda)^2+\beta^2 y^2}<|\lambda-\mu|\right\}$$

, then the $p+q$-th order partial derivative of $f(x,y)$,

$$\frac{\partial^{p+q}f}{\partial y^p\partial x^q}(x,y)$$

$$= \alpha^q\beta^p\sum_{k=0}^{\infty}\frac{(-n)_k(k-m)_{p+q}}{k!}(\lambda-\mu)^{-n-k}\times$$

$$[(\alpha x-\lambda)^2+\beta^2 y^2]^{(k-m-p-q)/2}\times$$

$$\cos\left((k-m-p-q)\cot^{-1}\frac{\alpha x-\lambda}{\beta y}+\frac{p\pi}{2}\right).$$

(7)

Proof Since $f(x,y)$

$$= \text{Re}\left[\frac{1}{[(\alpha x-\lambda)+i\beta y]^m\cdot[(\alpha x-\mu)+i\beta y]^n}\right]$$

(by Lemma 1)

$$= \text{Re}\left[\sum_{k=0}^{\infty}\frac{(-n)_k}{k!}(\lambda-\mu)^{-n-k}[(\alpha x-\lambda)+i\beta y]^{k-m}\right], \text{ (by}$$

Lemma 2)

it follows that

$$\frac{\partial^{p+q}f}{\partial y^p\partial x^q}(x,y)$$

$$= \alpha^q\beta^p\,\text{Re}\left[\sum_{k=0}^{\infty}\frac{(-n)_k(k-m)_{p+q}}{k!}i^p(\lambda-\mu)^{-n-k} \atop \times[(\alpha x-\lambda)+i\beta y]^{k-m-p-q}\right]$$

(by differentiation term by term theorem)

$$= \alpha^q\beta^p\sum_{k=0}^{\infty}\frac{(-n)_k(k-m)_{p+q}}{k!}(\lambda-\mu)^{-n-k}$$

$$\times[(\alpha x-\lambda)^2+\beta^2 y^2]^{(k-m-p-q)/2}$$

$$\times\cos\left((k-m-p-q)\cot^{-1}\frac{\alpha x-\lambda}{\beta y}+\frac{p\pi}{2}\right).$$

(by Lemma 3)

q.e.d.

The infinite series forms of any order partial derivatives of the two variables function (2) can be obtained below.

International Journal of Research

Available at https://edupediapublications.org/journals

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 04
February 2016

Theorem 2 *If the assumptions are the same as Theorem 1, and the domain of the two variables function*

$g(x, y)$

$$
=\frac{\left[\begin{array}{l} \sum_{s=0}^{m} \frac{(m)_s}{s!}(-\beta)^s \cos\left(\frac{s\pi}{2}\right)(\alpha x-\lambda)^{m-s} y^s \\ \times \sum_{t=0}^{n} \frac{(n)_t}{t!}(-\beta)^t \sin\left(\frac{t\pi}{2}\right)(\alpha x-\mu)^{n-t} y^t \\ + \sum_{s=0}^{m} \frac{(m)_s}{s!}(-\beta)^s \sin\left(\frac{s\pi}{2}\right)(\alpha x-\lambda)^{m-s} y^s \\ \times \sum_{t=0}^{n} \frac{(n)_t}{t!}(-\beta)^t \cos\left(\frac{t\pi}{2}\right)(\alpha x-\mu)^{n-t} y^t \end{array}\right]}{[(\alpha x-\lambda)^2 + \beta^2 y^2]^m \cdot [(\alpha x-\mu)^2 + \beta^2 y^2]^n}, is
$$

$\left\{(x, y) \in R^2 \middle| \beta y > 0, \sqrt{(\alpha x-\lambda)^2 + \beta^2 y^2} < |\lambda - \mu|\right\}$,

then the $p+q$-th order partial derivative of $g(x, y)$,

$$
\frac{\partial^{p+q} g}{\partial y^p \partial x^q}(x, y)
$$

$$
=\alpha^q \beta^p \sum_{k=0}^{\infty} \frac{(-n)_k (k-m)_{p+q}}{k!}(\lambda - \mu)^{-n-k}
$$

$$
\times [(\alpha x-\lambda)^2 + \beta^2 y^2]^{(k-m-p-q)/2}
$$

$$
\times \sin\left((k-m-p-q)\cot^{-1}\frac{\alpha x-\lambda}{\beta y} + \frac{p\pi}{2}\right).
$$

(8)

Proof Since $g(x, y)$

$$
=\text{Im}\left[\sum_{k=0}^{\infty} \frac{(-n)_k}{k!}(\lambda - \mu)^{-n-k}[(\alpha x-\lambda) + i\beta y]^{k-m}\right],
$$

using differentiation term by term theorem, we can easily obtain the desired result.

q.e.d.

3. Examples

For the partial differential problem of the two types of two variables functions in this paper, we will propose two examples and use Theorems 1 and 2 to determine the infinite

series forms of their any order partial derivatives. On the other hand, Maple is used to calculate the approximations of some higher order partial derivative values and their infinite series forms for verifying our answers.

Example 3.1 If the domain of the two variables function

$f(x, y)$

$$
=\frac{\left[\begin{array}{l} \sum_{s=0}^{3} \frac{(3)_s}{s!}(-4)^s \cos\left(\frac{s\pi}{2}\right)(2x-3)^{3-s} y^s \\ \times \sum_{t=0}^{2} \frac{(2)_t}{t!}(-4)^t \cos\left(\frac{t\pi}{2}\right)(2x-5)^{2-t} y^t \\ - \sum_{s=0}^{3} \frac{(3)_s}{s!}(-4)^s \sin\left(\frac{s\pi}{2}\right)(2x-3)^{3-s} y^s \\ \times \sum_{t=0}^{2} \frac{(2)_t}{t!}(-4)^t \sin\left(\frac{t\pi}{2}\right)(2x-5)^{2-t} y^t \end{array}\right]}{[(2x-3)^2 + 16y^2]^3 \cdot [(2x-5)^2 + 16y^2]^2}
$$

(9)

is $\left\{(x, y) \in R^2 \middle| y > 0, \sqrt{(2x-3)^2 + 16y^2} < 2\right\}$

(for $\alpha = 2, \beta = 4, \lambda = 3, \mu = 5, m = 3,$ and $n = 2$ in Theorem 1). Using Eq. (7) yields

$$
\frac{\partial^{p+q} f}{\partial y^p \partial x^q}(x, y)
$$

$$
=2^q 4^p \sum_{k=0}^{\infty} \frac{(-2)_k (k-3)_{p+q}}{k!}(-2)^{-2-k}
$$

$$
\times [(2x-3)^2 + 16y^2]^{(k-p-q-3)/2}
$$

$$
\times \cos\left((k-p-q-3)\cot^{-1}\frac{2x-3}{4y} + \frac{p\pi}{2}\right).
$$

(10)

Hence, the 10-th order partial derivative value of $f(x, y)$ at $\left(2, \frac{1}{8}\right)$,

International Journal of Research

Available at https://edupediapublications.org/journals

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 04
February 2016

$$\frac{\partial^{10} f}{\partial y^4 \partial x^6}\left(2, \frac{1}{8}\right)$$

$$= 16384 \sum_{k=0}^{\infty} \frac{(-2)_k (k-3)_{10}}{k!}(-2)^{-2-k}$$

$$\times \left(\frac{5}{4}\right)^{(k-13)/2} \cos[(k-13)\cot^{-1} 2].$$

(11)

Next, we use Maple to verify the correctness of Eq. (11).

```
>f:=(x,y)->(sum(product(3-j,j=0..(s-1))/s!*(-
4)^s*cos(s*Pi/2)*(2*x-3)^(3-s)*y^s,s=0..3)*
sum(product(2-j,j=0..(t-1))/t!*(-4)^t*cos(t*
Pi/2)*(2*x-5)^(2-t)*y^t,t=0..2)-sum(product
(3-j,j=0..(s-1))/s!*(-4)^s*sin(s*Pi/2)*(2*x-3)
^(3-s)*y^s,s=0..3)*sum(product(2-j,j=0..(t-
1))/t!*(-4)^t*sin(t*Pi/2)*(2*x-5)^(2-t)*y^t,t
=0..2))/(((2*x-3)^2+16*y^2)^3*((2*x-5)^2+
16*y^2)^2);
```

```
>evalf(D[1$6,2$4](f)(2,1/8),18);
```

$$2.73359754768612474 \cdot 10^{11}$$

```
>evalf(16384*sum(product(-2-j,j=0..(k-1))
*product(k-3-s,s=0..9)/k!*(-2)^(-2-k)*(5/4)^
((k-13)/2)*cos((k-13)*arccot(2)),k=0..
infinity),18);
```

$$2.73359754768612474 \cdot 10^{11}$$

Example 3.2 Suppose that the domain of the two variables function

$$g(x, y)$$

$$\frac{\left[\begin{array}{c} \sum_{s=0}^{2} \frac{(2)_s}{s!} 2^s \cos\left(\frac{s\pi}{2}\right)(6x+4)^{2-s} y^s \\ \times \sum_{t=0}^{4} \frac{(4)_t}{t!} 2^t \sin\left(\frac{t\pi}{2}\right)(6x-1)^{4-t} y^t \\ + \sum_{s=0}^{2} \frac{(2)_s}{s!} 2^s \sin\left(\frac{s\pi}{2}\right)(6x+4)^{2-s} y^s \\ \times \sum_{t=0}^{4} \frac{(4)_t}{t!} 2^t \cos\left(\frac{t\pi}{2}\right)(6x-1)^{4-t} y^t \end{array}\right]}{[(6x+4)^2 + 4y^2]^2 \cdot [(6x-1)^2 + 4y^2]^4}$$

(12)

is $\left\{(x, y) \in R^2 \middle| y < 0, \sqrt{(6x+4)^2 + 4y^2} < 5\right\}$

(for $\alpha = 6, \beta = -2, \lambda = -4, \mu = 1, m = 2$, and $n = 4$ in Theorem 2). By Eq. (8) we have

$$\frac{\partial^{p+q} g}{\partial y^p \partial x^q}(x, y)$$

$$= 6^q (-2)^p \sum_{k=0}^{\infty} \frac{(-4)_k (k-2)_{p+q}}{k!}(-5)^{-4-k}$$

$$\times [(6x+4)^2 + 4y^2]^{(k-p-q-2)/2}$$

$$\times \sin\left((k-p-q-2)\cot^{-1}\frac{6x+4}{-2y} + \frac{p\pi}{2}\right).$$

(13)

Thus, the 12-th order partial derivative value of

$g(x, y)$ at $\left(-\frac{1}{2}, -\frac{1}{4}\right)$,

$$\frac{\partial^{12} g}{\partial y^8 \partial x^4}\left(-\frac{1}{2}, -\frac{1}{4}\right)$$

$$= 331776 \sum_{k=0}^{\infty} \frac{(-4)_k (k-2)_{12}}{k!}(-5)^{-4-k}$$

$$\times \left(\frac{5}{4}\right)^{(k-14)/2} \sin[(k-14)\cot^{-1} 2].$$

(14)

Using Maple to verify the correctness of Eq. (14) as follows:

```
>g:=(x,y)->(sum(product(2-j,j=0..(s-1))/s!*2
```

^s*cos(s*Pi/2)*(6*x+4)^(2-s)*y^s,s=0..2)*sum(product(4-j,j=0..(t-1))/t!*2^t*sin(t*Pi/2)*(6*x-1)^(4-t)*y^t,t=0..4)+sum(product(2-j,j=0..(s-1))/s!*2^s*sin(s*Pi/2)*(6*x+4)^(2-s)*y^s,s=0..2)*sum(product(4-j,j=0..(t-1))/t!*2^t*cos(t*Pi/2)*(6*x-1)^(4-t)*y^t,t=0..4))/(((6*x+4)^2+4*y^2)^2*((6*x-1)^2+4*y^2)^4);

>evalf(D[1$4,2$8](g)(-1/2,-1/4),18);

$$-1.31005997719218872 \cdot 10^{11}$$

>evalf(331776*sum(product(-4-j,j=0..(k-1))*product(k-2-s,s=0..11)/k!*(-5)^(-4-k)*(5/4)^((k-14)/2)*sin((k-14)*arccot(2)),k=0..infinity),18);

$$-1.31005997719218872 \cdot 10^{11}$$

4. Conclusion

As mentioned, the evaluation and numerical calculation of the partial derivatives of multivariable functions are important in calculus and engineering mathematics. In this article, we mainly use binomial series and differentiation term by term theorem to evaluate the partial derivatives of two-variables functions. In fact, the applications of the two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems using Maple.

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