Complex Analysis Method for Evaluating the Partial Derivatives of Two Variables Functions

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Abstract

This paper uses the mathematical software Maple for the auxiliary tool to study the partial differential problem of two types of two variables functions. We can obtain the infinite series forms of any order partial derivatives of these two types of functions mainly using complex analysis method and differentiation term by term theorem. In addition, we provide some examples to do calculation practically. The research methods adopted in this study is to find the solutions through manual calculations and verify these solutions using Maple.

Key Words: partial derivatives; infinite series forms; complex analysis method; differentiation term by term theorem; Maple

1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics, the evaluation and numerical calculation of the partial derivatives of multivariable functions are important. The Laplace equation, the wave equation, and other important physical equations involve the partial derivatives. The evaluation of the t-th order partial derivative value of a multivariable function at some point, generally, requires two procedures: the determination of the t-th order partial derivative of the function, and the substitution of the point into the t-th order partial derivative. These two procedures become increasingly complex calculations for increasing order of partial derivative, and thus manual calculations become difficult.

The present paper considers the partial differential problem of the following two types of two variables functions

$$f(r,\theta)=\exp(ar\cos\theta)\times [\cos(ar\sin\theta)\cos(br\cos\theta)\cosh(br\sin\theta)+ \sin(ar\sin\theta)\sin(br\cos\theta)\sinh(br\sin\theta)], \quad (1)$$

$$g(r,\theta)=\exp(ar\cos\theta)\times [-\cos(ar\sin\theta)\sin(br\cos\theta)\sinh(br\sin\theta)+ \sin(ar\sin\theta)\cos(br\cos\theta)\cosh(br\sin\theta)], \quad (2)$$

where a,b,r,θ are real numbers. The infinite series forms of any order partial derivatives of these two types of functions can be obtained by using complex analysis method and differentiation term by term theorem, these are the major results in this paper: Theorems 1 and 2. The study of related partial differential problems can refer to [1-20]. [1-5] used some methods to evaluate the partial derivatives of multivariable functions, which are different from the methods used in this paper, and [6-20] took advantage of some techniques, for example, complex power series, binomial series and differentiation term by term theorem to study the partial differential problem. On the other hand, we propose two examples of two variables functions to evaluate their any order partial derivatives, and some of their higher order partial derivative values practically. In addition, Maple is used to calculate the approximations of these higher order partial derivative values and their infinite series forms for verifying our answers.

2. Preliminaries and Main Results

Some notations, formulas, and theorems used in this paper are introduced below.

2.1 Notations:

2.1.1 Let $z=a+ib$ be a complex number, where $i=\sqrt{-1}$, and a,b are real numbers. a , the real part of z , is denoted as $\operatorname{Re}(z)$; b , the imaginary part of z , is denoted as $\operatorname{Im}(z)$.

2.1.2 Assume that r is a real number, s is a positive integer, and $s\le r$. We define $(r)_s=r(r-1)\cdots(r-s+1)$, and $(r)_0=1$.

2.1.3 Suppose that p,q are non-negative integers. For the two variables function $f(r,\theta)$, its q -times partial derivative with respect to r , and p -times partial derivative with respect to θ , forms an $p+q$ -th order partial derivative, and is denoted as $\frac{\partial^{p+q}f}{\partial\theta^p\partial r^q}(r,\theta)$.

2.2 Formulas and theorems:

2.2.1 Euler's formula:
$e^{i\theta}=\cos\theta+i\sin\theta$, where θ is a real number.

2.2.2 DeMoivre's formula:
$(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$, where m is an integer, and θ is a real number.

2.2.3 Differentiation term by term theorem ([21, p230]):

If, for all non-negative integer k , the functions $g_k:(a,b)\to R$ satisfy the following three conditions : (i) there exists a point $x_0\in(a,b)$ such that $\sum_{k=0}^{\infty}g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b) , (iii)

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$\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b). Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b). Moreover, $\frac{d}{dx}\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

Before the major results can be derived, we need the following two lemmas.

Lemma 1 *Suppose that* a,b *are real numbers, and* n *is a positive integer, then*

$$\mathrm{Re}[(a+ib)^n] = \sum_{k=0}^{n} \frac{(n)_k}{k!} a^{n-k} b^k \cos\frac{k\pi}{2}. \tag{3}$$

Proof $\mathrm{Re}[(a+ib)^n]$

$$= \mathrm{Re}\left[\sum_{k=0}^{n} \frac{(n)_k}{k!} a^{n-k} (ib)^k\right]$$

$$= \mathrm{Re}\left[\sum_{k=0}^{n} \frac{(n)_k}{k!} a^{n-k} b^k \exp(ik\pi/2)\right]$$

(by Euler's formula and DeMoivre's formula)

$$= \sum_{k=0}^{n} \frac{(n)_k}{k!} a^{n-k} b^k \cos\frac{k\pi}{2}. \qquad \text{q.e.d.}$$

Lemma 2 *If* a,b *are real numbers, and* z *is a complex number, then*

$$\exp(az)\cos bz = \sum_{n=0}^{\infty} \frac{\mathrm{Re}[(a+ib)^n]}{n!} z^n. \tag{4}$$

Proof $\exp(az)\cos bz$

$$= \exp(az)\cdot\frac{1}{2}[\exp(ibz)+\exp(-ibz)]$$

$$= \frac{1}{2}\{\exp[(a+ib)z]+\exp[(a-ib)z]\}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(a+ib)^n}{n!} z^n + \sum_{n=0}^{\infty} \frac{(a-ib)^n}{n!} z^n\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{[(a+ib)^n+(a-ib)^n]}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{\mathrm{Re}[(a+ib)^n]}{n!} z^n. \qquad \text{q.e.d.}$$

In the following, we determine the infinite series forms of any order partial derivatives of the two variables function (1).

Theorem 1 *Suppose that* a,b *are real numbers, and* p,q *are non-negative integers. If the domain of the two variables function*

$$f(r,\theta) = \exp(ar\cos\theta)\times$$
$$[\cos(ar\sin\theta)\cos(br\cos\theta)\cosh(br\sin\theta)+ \sin(ar\sin\theta)\sin(br\cos\theta)\sinh(br\sin\theta)]$$

is R^2, *then the* $p+q$ *-th order partial derivative of* $f(r,\theta)$,

$$\frac{\partial^{p+q} f}{\partial\theta^p \partial r^q}(r,\theta)$$
$$= \sum_{n=0}^{\infty} \frac{n^p (n)_q \sum_{k=0}^{n} \frac{(n)_k a^{n-k} b^k \cos\frac{k\pi}{2}}{k!}}{n!} r^{n-q} \cos\left(n\theta+\frac{p\pi}{2}\right) \tag{5}$$

for all $(r,\theta)\in R^2$.

Proof Since $f(r,\theta)$

$$= \mathrm{Re}\{\exp[ar\exp(i\theta)]\cdot\cos[br\exp(i\theta)]\}$$

$$= \mathrm{Re}\left\{\sum_{n=0}^{\infty} \frac{\mathrm{Re}[(a+ib)^n]}{n!}[r\exp(i\theta)]^n\right\}$$

(by Eq.(5))

$$= \sum_{n=0}^{\infty} \frac{\mathrm{Re}[(a+ib)^n]}{n!} r^n \cos n\theta$$

(by Euler's formula and DeMoivre's formula)

$$=\sum_{n=0}^{\infty}\frac{\sum_{k=0}^{n}\frac{(n)_k}{k!}a^{n-k}b^k\cos\frac{k\pi}{2}}{n!}r^n\cos n\theta,$$

(by Eq. (3))

we differentiate q -times with respect to r, and p -times with respect to θ for $f(r,\theta)$, and using differentiation term by term theorem yields the desired result holds.

q.e.d.

The infinite series forms of any order partial derivatives of the two variables function (2) can be obtained below.

Theorem 2 *If the assumptions are the same as Theorem 1, and the domain of the two variables function*

$$g(r,\theta)=\exp(ar\cos\theta)\times$$
$$[-\cos(ar\sin\theta)\sin(br\cos\theta)\sinh(br\sin\theta)+\sin(ar\sin\theta)\cos(br\cos\theta)\cosh(br\sin\theta)]$$

is R^2, *then the* $p+q$ *-th order partial derivative of* $g(r,\theta)$,

$$\frac{\partial^{p+q}g}{\partial\theta^p\partial r^q}(r,\theta)=\sum_{n=0}^{\infty}\frac{n^p(n)_q\sum_{k=0}^{n}\frac{(n)_k a^{n-k}b^k\cos\frac{k\pi}{2}}{k!}}{n!}r^{n-q}\sin\left(n\theta+\frac{p\pi}{2}\right) \quad (6)$$

for all $(r,\theta)\in R^2$.

Proof Since $g(r,\theta)$

$$=\mathrm{Im}\{\exp[ar\exp(i\theta)]\cdot\cos[br\exp(i\theta)]\}$$

$$=\mathrm{Im}\left\{\sum_{n=0}^{\infty}\frac{\mathrm{Re}[(a+ib)^n]}{n!}[r\exp(i\theta)]^n\right\}$$

$$=\sum_{n=0}^{\infty}\frac{\mathrm{Re}[(a+ib)^n]}{n!}r^n\sin n\theta$$

$$=\sum_{n=0}^{\infty}\frac{\sum_{k=0}^{n}\frac{(n)_k}{k!}a^{n-k}b^k\cos\frac{k\pi}{2}}{n!}r^n\sin n\theta,$$

also by differentiation term by term theorem, we obtain the desired result.

q.e.d.

3. Examples

In the following, for the partial differential problem of the two types of two variables functions discussed in this paper, some examples are proposed and we use Theorems 1 and 2 to obtain the infinite series forms of their any order partial derivatives. In addition, Maple is used to calculate the approximations of some partial derivatives values and their infinite series forms for verifying our answers.

Example 1 If the domain of the two variables function

$$f(r,\theta)=\exp(2r\cos\theta)\times$$
$$[\cos(2r\sin\theta)\cos(5r\cos\theta)\cosh(5r\sin\theta)+\sin(2r\sin\theta)\sin(5r\cos\theta)\sinh(5r\sin\theta)] \quad (7)$$

is R^2 (for $a=2, b=5$ in Theorem 1), then using Eq. (5) yields

$$\frac{\partial^{p+q}f}{\partial\theta^p\partial r^q}(r,\theta)=\sum_{n=0}^{\infty}\frac{n^p(n)_q\sum_{k=0}^{n}\frac{(n)_k 2^{n-k}5^k\cos\frac{k\pi}{2}}{k!}}{n!}r^{n-q}\cos\left(n\theta+\frac{p\pi}{2}\right), \quad (8)$$

for all $(r,\theta)\in R^2$.

Thus,

$$\frac{\partial^7 f}{\partial\theta^4\partial r^3}\left(8,\frac{\pi}{3}\right)$$

$$=\sum_{n=0}^{\infty}\frac{n^4(n)_3\sum_{k=0}^{n}\frac{(n)_k 2^{n-k}5^k\cos\frac{k\pi}{2}}{k!}}{n!}8^{n-3}\cos\frac{n\pi}{3}.$$

(9)

Next, we use Maple to verify the correctness of Eq. (9).

```
>f:=(r,theta)->exp(2*r*cos(theta))*(cos(2*r
*sin(theta))*cos(5*r*cos(theta))*cosh(5*r*
sin(theta))+sin(2*r*sin(theta))*sin(5*r*cos(
theta))*sinh(5*r*sin(theta)));
>evalf(D[1$3,2$4](f)(8,Pi/3),30);
```

$8.98017774156654264450745073339\cdot 10^{26}$

```
>evalf(sum(n^4*product(n-s,s=0..2)*sum(
product(n-t,t=0..(k-1))*2^(n-k)*5^k*cos(k*
Pi/2)/k!,k=0..n)/n!*8^(n-3)*cos(n*Pi/3),n=0..
infinity),30);
```

$8.98017774156654264450745073209\cdot 10^{26}$

Example 2 Let the domain of the two variables function

$$g(r,\theta)=\exp(3r\cos\theta)\times$$
$$[-\cos(3r\sin\theta)\sin(2r\cos\theta)\sinh(2r\sin\theta)+$$
$$\sin(3r\sin\theta)\cos(2r\cos\theta)\cosh(2r\sin\theta)],$$

(10)

be R^2 (for $a=3, b=2$ in Theorem 2), then by Eq. (6) we have

$$\frac{\partial^{p+q}g}{\partial\theta^p\partial r^q}(r,\theta)$$
$$=\sum_{n=0}^{\infty}\frac{n^p(n)_q\sum_{k=0}^{n}\frac{(n)_k 3^{n-k}2^k\cos\frac{k\pi}{2}}{k!}}{n!}r^{n-q}\sin\left(n\theta+\frac{p\pi}{2}\right),$$

(11)

for all $(r,\theta)\in R^2$.

Therefore,

$$\frac{\partial^{11}g}{\partial\theta^6\partial r^5}\left(3,\frac{\pi}{4}\right)$$
$$=-\sum_{n=0}^{\infty}\frac{n^6(n)_5\sum_{k=0}^{n}\frac{(n)_k 3^{n-k}2^k\cos\frac{k\pi}{2}}{k!}}{n!}3^{n-5}\sin\frac{n\pi}{4}.$$

(12)

Using Maple to verify the correctness of Eq. (12) as follows:

```
>g:=(r,theta)->exp(3*r*cos(theta))*(-cos(3*
r*sin(theta))*sin(2*r*cos(theta))*sinh(2*r*
sin(theta))+sin(3*r*sin(theta))*cos(2*r*cos(the
ta))*cosh(2*r*sin(theta)));
>evalf(D[1$5,2$6](g)(3,Pi/4),30);
```

$2.27490689771922447005831592254\cdot 10^{14}$

```
>evalf(-sum(n^6*product(n-s,s=0..4)*sum(
product(n-t,t=0..(k-1))*3^(n-k)*2^k*cos(k*
Pi/2)/k!,k=0..n)/n!*3^(n-5)*sin(n*Pi/4),n=0
..infinity),30);
```

$2.27490689771922447005831592255\cdot 10^{14}$

4. Conclusion

The evaluation and numerical calculation of the partial derivatives of multivariable functions are important in calculus and engineering mathematics. In this study, we use differentiation term by term theorem to determine any order partial derivatives of two types of two variables functions. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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