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Using Laplace Transform to Evaluate Improper Integrals

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Abstract

In this article, we use Maple for the auxiliary tool to study four types of improper integrals. The closed forms of these improper integrals can be obtained by using Laplace transform. On the other hand, some examples are proposed to demonstrate the calculations, and we verify their answers using Maple.

Key Words: improper integrals; closed forms; Laplace transform; Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests.

In calculus and engineering mathematics courses, there are many methods to solve the integral problems, for example, change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. This paper studies the following four types of improper integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^\infty t^n e^{-xt} \cos(yt)dt, \tag{1}$$

$$\int_0^\infty t^n e^{-xt} \sin(yt)dt, \tag{2}$$

$$\int_0^\infty t^n e^{-(r\cos\theta)t} \cos[(r\sin\theta)t]dt, \tag{3}$$

$$\int_0^\infty t^n e^{-(r\cos\theta)t} \sin[(r\sin\theta)t]dt, \tag{4}$$

where x, y, r, θ are real numbers, n is a non-negative integer, $x > 0$, and $r\cos\theta > 0$. The closed forms of these four types of improper integrals can be obtained by using Laplace transform, these are the major results in this paper: Theorems 1 and 2. Adams et al. [1], Nyblom [2], and Oster [3] provided some methods to solve some integral problems. Yu [4-24], Yu and Chen [25], and Yu and Sheu [26-28] used complex power series, complex integral formulas, integration term by term, differentiation with respect to a parameter, Parseval's theorem, and area mean value theorem to solve some types of integrals. In this article, we propose some improper integrals to do calculation practically. On the other hand, Maple is used to calculate the approximations of these improper integrals and their closed forms for verifying our answers.

2. Preliminaries and Results

Some notations and formulas used in this paper are introduced below.

2.1 Notations:

2.1.1 Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, and a, b are real numbers. a, the real part of z, is denoted as $\operatorname{Re}(z)$; b, the imaginary part of z, is denoted as $\operatorname{Im}(z)$.

2.1.2 $(s)_n = s(s-1)\cdots(s-n+1)$, whers s is a real number, and n is a positive integer; $(s)_0 = 1$.

2.2 Formulas:

2.2.1 Euler's formula:
$e^{i\theta} = \cos\theta + i\sin\theta$, where θ is any real number.

2.2.2 DeMoivre's formula:
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is an integer, and θ is a real number.

2.2.3 Laplace transform ([29, p607]):

Suppose that n is a non-negative integer, and s is a complex number with $\operatorname{Re}(s) > 0$, then

$$\int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}}. \tag{5}$$

In the following, we determine the closed forms of the improper integrals (1) and (2).

Theorem 1 *Suppose that* x, y *are real numbers,* n *is a non-negative integer, and* $x > 0$, *then*

$$\int_0^\infty t^n e^{-xt} \cos(yt) dt = \frac{n! \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)_k}{k!} \cos\frac{k\pi}{2} \cdot x^{n-k+1} y^k}{(x^2+y^2)^{n+1}}, \tag{6}$$

and

$$\int_0^\infty t^n e^{-xt} \sin(yt) dt = \frac{-n! \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)_k}{k!} \sin\frac{k\pi}{2} \cdot x^{n-k+1} y^k}{(x^2+y^2)^{n+1}}. \tag{7}$$

Proof Using Eq. (5) yields

$$\int_0^\infty t^n e^{-(x+iy)t} dt = \frac{n!}{(x+iy)^{n+1}}. \tag{8}$$

Thus,

$$\int_0^\infty t^n e^{-xt} \cos(yt) dt = \operatorname{Re}\left[\frac{n!}{(x+iy)^{n+1}}\right]$$

(by Euler's formula)

$$= \frac{n! \operatorname{Re}[(x-iy)^{n+1}]}{(x^2+y^2)^{n+1}}$$

$$= \frac{n! \operatorname{Re}\left[\sum_{k=0}^{n+1} \frac{(n+1)_k}{k!} x^{n-k+1} (-iy)^k\right]}{(x^2+y^2)^{n+1}}$$

$$= \frac{n! \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)_k}{k!} \cos\frac{k\pi}{2} \cdot x^{n-k+1} y^k}{(x^2+y^2)^{n+1}}.$$

Similarly, by Eq. (8) we have

$$\int_0^\infty t^n e^{-xt} \sin(yt) dt = -\operatorname{Im}\left[\frac{n!}{(x+iy)^{n+1}}\right]$$

$$= \frac{-n! \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)_k}{k!} \sin\frac{k\pi}{2} \cdot x^{n-k+1} y^k}{(x^2+y^2)^{n+1}}.$$

q.e.d.

The closed forms of the improper integrals (3) and (4) can be obtained below.

Theorem 2 *If* r, θ *are real numbers,* n *is a non-negative integer, and* $r\cos\theta > 0$*, then*

$$\int_0^\infty t^n e^{-(r\cos\theta)t}\cos[(r\sin\theta)t]dt = \frac{n!\cos[(n+1)\theta]}{r^{n+1}}, \quad (9)$$

and

$$\int_0^\infty t^n e^{-(r\cos\theta)t}\sin[(r\sin\theta)t]dt = \frac{n!\sin[(n+1)\theta]}{r^{n+1}}. \quad (10)$$

Proof Using Eq. (5) yields

$$\int_0^\infty t^n e^{-re^{i\theta}t}dt = \frac{n!}{(re^{i\theta})^{n+1}}. \quad (11)$$

Hence,

$$\int_0^\infty t^n e^{-(r\cos\theta)t}\cos[(r\sin\theta)t]dt$$

$$= \mathrm{Re}\left[\frac{n!}{(re^{i\theta})^{n+1}}\right]$$

(by Euler's formula)

$$= \mathrm{Re}\left[\frac{n!}{r^{n+1}e^{i(n+1)\theta}}\right]$$

(by DeMoivre's formula)

$$= \frac{n!\cos[(n+1)\theta]}{r^{n+1}}.$$

Also, using Eq. (11) yields

$$\int_0^\infty t^n e^{-(r\cos\theta)t}\sin[(r\sin\theta)t]dt$$

$$= -\mathrm{Im}\left[\frac{n!}{(re^{i\theta})^{n+1}}\right]$$

$$= \frac{n!\sin[(n+1)\theta]}{r^{n+1}}. \quad \text{q.e.d.}$$

3. Example

For the four types of improper integrals in this study, we will propose some examples and use Theorems 1 and 2 to obtain their closed forms. In addition, Maple is used to calculate the approximations of these improper integrals and their closed forms for verifying our answers.

Example 1 By Eq. (6), we have

$$\int_0^\infty t^3 e^{-2t}\cos(5t)dt$$

$$= \frac{6\sum_{k=0}^{4}\frac{(-1)^k(4)_k}{k!}\cos\frac{k\pi}{2}\cdot 2^{4-k}5^k}{29^4}$$

$$= \frac{246}{707281}. \quad (12)$$

The correctness of Eq. (12) can be verified by using Maple.

```
>evalf(int(t^3*exp(-2*t)*cos(5*t),t=0..infinity),18);
```

0.000347810841801207724

```
>evalf(246/707281,18);
```

0.000347810841801207724

On the other hand, using Eq. (7) yields

$$\int_0^\infty t^6 e^{-4t}\sin(7t)dt$$

$$= \frac{-6!\sum_{k=0}^{7}\frac{(-1)^k(7)_k}{k!}\sin\frac{k\pi}{2}\cdot 4^{7-k}7^k}{65^7}$$

$$= \frac{280948752}{980445578125}. \quad (13)$$

We also use Maple to verify the correctness of Eq. (13).

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```
>evalf(int(t^6*exp(-4*t)*sin(7*t),t=0..
infinity),18);
```

0.000286552112904915346

```
>evalf(280948752/980445578125,18);
```

0.000286552112904915346

Example 2 We can determine the following improper integral using Eq. (9),

$$\int_0^\infty t^4 e^{-[2\cos(\pi/8)]t} \cos[2\sin(\pi/8)t]dt = \frac{3\cos(5\pi/8)}{4}. \qquad (14)$$

The correctness of Eq. (14) can be verified by using Maple.

```
>evalf(int(t^4*exp(-2*cos(Pi/8)*t)*cos(2*
sin(Pi/8)*t),t=0..infinity),18);
```

−0.287012574273817332

```
>evalf(3*cos(5*Pi/8)/4,18);
```

−0.287012574273817332

In addition, using Eq.(10) yields

$$\int_0^\infty t^7 e^{-[3\cos(\pi/5)]t} \sin[3\sin(\pi/5)t]dt$$

$$= \frac{560\sin(8\pi/5)}{729}. \qquad (15)$$

```
>evalf(int(t^7*exp(-3*cos(Pi/5)*t)*sin(3*
sin(Pi/5)*t),t=0..infinity),18);
```

−0.730578393861846366

```
>evalf(560*sin(8*Pi/5)/729,18);
```

−0.730578393861846367

4. Conclusion

In this paper, we use Laplace transform to solve some improper integrals. In fact, the applications of Laplace transform are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topics to other calculus and engineering mathematics problems and solve these problems using Maple.

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