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Studying the Problems of Multiple Integrals with Maple

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Abstract

In this article, we use Maple for the auxiliary tool to study two types of multiple integrals. The infinite series forms of the two types of multiple integrals can be obtained mainly using binomial series and integration term by term theorem. Moreover, some examples are proposed to demonstrate the calculations, and we use Maple to calculate the approximations of some multiple integrals and their infinite series forms for verifying our answers.

Key Words: Maple; multiple integrals; infinite series forms; binomial series; integration term by term theorem

1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

The multiple integral problem is closely related with probability theory and quantum field theory, and can be studied based on Streit [1] and Ryder [2]. For this reason, the evaluation and numerical calculation of multiple integrals is important. For the study of related multiple integral problems can refer to Yu [3-10]. In this paper, we mainly study the following two types of n-tuple integrals

$$\int_0^1 \cdots \int_0^1 \frac{\prod_{i=1}^{n} x_i^{q_i}}{\left(\prod_{i=1}^{n} x_i^{p_i} + a\right)^r} dx_1 \cdots dx_n, \quad (1)$$

and

$$\int_1^\infty \cdots \int_1^\infty \frac{\prod_{i=1}^{n} x_i^{q_i}}{\left(\prod_{i=1}^{n} x_i^{p_i} + a\right)^r} dx_1 \cdots dx_n, \quad (2)$$

where a, r are real numbers, $r > 0$, n is a positive integer, p_i, q_i are non-negative integers for all $i = 1, \cdots, n$, and some conditions are satisfied. We can determine the infinite series forms of the two types of multiple integrals mainly using binomial series and integration term by term theorem; these are the major results of this article (i.e., Theorems 1 and 2). In addition, two examples are used to demonstrate the proposed calculations. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Preliminaries and Main Results

First, a notation and two important theorems used in this study are introduced below.

2.1 Notation: $\prod_{i=1}^{n} x_i^{\,p_i} = x_1^{p_1} \cdot x_2^{p_2} \cdots x_n^{p_n}$.

2.2 Binomial series ([11,p244])：Suppose that a, x are real numbers and $|x| < 1$, then

$$(1+x)^a = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} x^k ,$$

where $(a)_k = a(a-1)\cdots(a-k+1)$

for all positive integers k , and $(a)_0 = 1$.

2.2 Integration term by term theorem ([11, p269]): Suppose that $\{g_n\}_{n=0}^{\infty}$ is a sequence of Lebesgue integrable functions defined on an inteval I . If $\sum_{n=0}^{\infty} \int_I |g_n|$ is convergent, then $\int_I \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} \int_I g_n$.

Before deriving the major results in this paper, a lemma is needed.

Lemma *Assume that* a, r *are real numbers,* $r > 0$, n *is a positive integer, and* p_i, q_i *are non-negative integers for all* $i = 1, \cdots, n$.

Case 1. If $\left|\prod_{i=1}^{n} x_i^{\,p_i}\right| < |a|$ *, then*

$$\frac{\prod_{i=1}^{n} x_i^{\,q_i}}{\left(\prod_{i=1}^{n} x_i^{\,p_i} + a\right)^r} = \sum_{k=0}^{\infty} \frac{(-1)^k (k+r-1)_k}{k!\, a^{k+r}} \prod_{i=1}^{n} x_i^{\,p_i k + q_i} . \tag{3}$$

Case 2. If $\left|\prod_{i=1}^{n} x_i^{\,p_i}\right| > |a|$ *, then*

$$\frac{\prod_{i=1}^{n} x_i^{\,q_i}}{\left(\prod_{i=1}^{n} x_i^{\,p_i} + a\right)^r} = \sum_{k=0}^{\infty} \frac{(-a)^k (k+r-1)_k}{k!} \prod_{i=1}^{n} x_i^{\,q_i - p_i k - p_i r} . \tag{4}$$

Proof If $|x| < 1$, then by binomial series we have

$$\frac{1}{(x+1)^r} = \sum_{k=0}^{\infty} \frac{(-1)^k (k+r-1)_k}{k!} x^k . \tag{5}$$

Case 1. If $\left|\prod_{i=1}^{n} x_i^{\,p_i}\right| < |a|$, then

$$\frac{\prod_{i=1}^{n} x_i^{\,q_i}}{\left(\prod_{i=1}^{n} x_i^{\,p_i} + a\right)^r}$$

$$=\frac{\prod_{i=1}^{n} x_i^{q_i}}{a^r}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k(k+r-1)_k}{k!}\left(\frac{1}{a}\prod_{i=1}^{n} x_i^{p_i}\right)^k$$

(byEq. (5))

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(k+r-1)_k}{k!a^{k+r}}\prod_{i=1}^{n} x_i^{p_ik+q_i}.$$

Case 2. If $\left|\prod_{i=1}^{n} x_i^{p_i}\right|>|a|$, then

$$\frac{\prod_{i=1}^{n} x_i^{q_i}}{\left(\prod_{i=1}^{n} x_i^{p_i}+a\right)^r}$$

$$=\frac{\prod_{i=1}^{n} x_i^{q_i}}{\left(\prod_{i=1}^{n} x_i^{p_i}\right)^r}\cdot\left(\frac{a}{\prod_{i=1}^{n} x_i^{p_i}}+1\right)^{-r}$$

$$=\sum_{k=0}^{\infty}\frac{(-a)^k(k+r-1)_k}{k!}\prod_{i=1}^{n} x_i^{q_i-p_ik-p_ir}.$$

(byEq. (5)) q.e.d.

The following is the first result in this article, we obtain the infinite series form of the multiple integral (1).

Theorem 1 *If* a, r *are real numbers,* $|a|>1$, $r>0$, n *is a positive integer, and* p_i, q_i *are non-negative integers for all* $i=1,\cdots,n$, *then*

$$\int_0^1\cdots\int_0^1\frac{\prod_{i=1}^{n} x_i^{q_i}}{\left(\prod_{i=1}^{n} x_i^{p_i}+a\right)^r}dx_1\cdots dx_n$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(k+r-1)_k}{k!a^{k+r}\prod_{i=1}^{n}(p_ik+q_i+1)}. \quad (6)$$

Proof Since $|a|>1$, it follows that $\left|\prod_{i=1}^{n} x_i^{p_i}\right|<|a|$. Therefore,

$$\int_0^1\cdots\int_0^1\frac{\prod_{i=1}^{n} x_i^{q_i}}{\left(\prod_{i=1}^{n} x_i^{p_i}+a\right)^r}dx_1\cdots dx_n$$

$$=\int_0^1\cdots\int_0^1\sum_{k=0}^{\infty}\frac{(-1)^k(k+r-1)_k}{k!a^{k+r}}\prod_{i=1}^{n} x_i^{p_ik+q_i}dx_1\cdots dx_n$$

(by Eq. (3))

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(k+r-1)_k}{k!a^{k+r}}\int_0^1\cdots\int_0^1\prod_{i=1}^{n} x_i^{p_ik+q_i}dx_1\cdots dx_n$$

(by integration term by term theorem)

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(k+r-1)_k}{k!a^{k+r}}\prod_{i=1}^{n}\int_0^1 x_i^{p_ik+q_i}dx_i$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(k+r-1)_k}{k!a^{k+r}\prod_{i=1}^{n}(p_ik+q_i+1)}.$$

q.e.d.

Next, we determine the infinite series form of the multiple integral (2).

Theorem 2 *Suppose that* a, r *are real numbers,* $|a|<1$, $r>0$, n *is a positive integer, and* p_i, q_i *are non-negative integers such that* $q_i-p_ir+1<0$ *for all* $i=1,\cdots,n$, *then*

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$$\int_1^\infty \cdots \int_1^\infty \frac{\prod_{i=1}^n x_i^{q_i}}{\left(\prod_{i=1}^n x_i^{p_i}+a\right)^r} dx_1 \cdots dx_n$$

$$=(-1)^n \sum_{k=0}^\infty \frac{(-a)^k (k+r-1)_k}{k!\prod_{i=1}^n (q_i - p_i k - p_i r + 1)}. \quad (7)$$

Proof $|a|<1$ implies that $\left|\prod_{i=1}^n x_i^{p_i}\right| > |a|$. Thus,

$$\int_1^\infty \cdots \int_1^\infty \frac{\prod_{i=1}^n x_i^{q_i}}{\left(\prod_{i=1}^n x_i^{p_i}+a\right)^r} dx_1 \cdots dx_n$$

$$=\int_1^\infty \cdots \int_1^\infty \sum_{k=0}^\infty \frac{(-a)^k (k+r-1)_k}{k!} \prod_{i=1}^n x_i^{q_i - p_i k - p_i r} dx_1 \cdots dx_n$$

(by Eq. (4))

$$=\sum_{k=0}^\infty \frac{(-a)^k (k+r-1)_k}{k!} \int_1^\infty \cdots \int_1^\infty \prod_{i=1}^n x_i^{q_i - p_i k - p_i r} dx_1 \cdots dx_n$$ (by integration term by term theorem)

$$=\sum_{k=0}^\infty \frac{(-a)^k (k+r-1)_k}{k!} \prod_{i=1}^n \int_1^\infty x_i^{q_i - p_i k - p_i r} dx_i$$

$$=(-1)^n \sum_{k=0}^\infty \frac{(-a)^k (k+r-1)_k}{k!\prod_{i=1}^n (q_i - p_i k - p_i r + 1)}.$$

q.e.d.

3. Examples

For the multiple integral problems discussed in this paper, we propose two examples and use Theorems 1 and 2 to obtain their infinite series forms. Moreover, Maple is used to calculate the approximations of some multiple integrals and their solutions to verify our answers.

Example 1 Using Eq. (6) in Theorem 1 yields the double integral

$$\int_0^1 \int_0^1 \frac{x_1^5 x_2^3}{(x_1^2 x_2^4 + 3)^{7/6}} dx_1 dx_2$$

$$=\sum_{k=0}^\infty \frac{(-1)^k (k+1/6)_k}{k! 3^{k+7/6} (2k+6)(4k+4)}. \quad (8)$$

Next, we use Maple to verify the correctness of Eq. (8).

```
>evalf(Doubleint(x1^5*x2^3/(x1^2*x2^4+3)^(7/6),x1=0..1,x2=0..1),14);
```

0.010145480892988

```
>evalf(sum((-1)^k*product(k+1/6-j,j=0..(k-1))/(k!*3^(k+7/6)*(2*k+6)*(4*k+4)),k=0..infinity),16);
```

0.010145480892986

Example 2 On the other hand, by Eq. (7) in Theorem 2, we obtain the following triple improper integral

$$\int_1^\infty \int_1^\infty \int_1^\infty \frac{x_1^2 x_2^3 x_3^4}{(x_1^6 x_2^2 x_3^5 + 3/4)^{9/2}} dx_1 dx_2 dx_3$$

$$=-\sum_{k=0}^\infty \frac{(-3/4)^k (k+7/2)_k}{k!(-6k-24)(-2k-5)(-5k-35/2)}. \quad (9)$$

We also employ Maple to verify the correctness of Eq. (9).

```
>evalf(Tripleint(x1^2*x2^3*x3^4/(x1^6*x2^2*x3^5+3/4)^(9/2),x1=1..infinity,x2=1..infinity,x3=1..infinity),14);
```

0.00015150456630371

```
>evalf(-sum((-3/4)^k*product(k+7/2-j,j=0..(k-1))/(k!*(-6*k-24)*(-2*k-5)*(-5*k-35/2)),k=0..infinity),16);
```

0.00015150456630369

4. Conclusion

As mentioned, the evaluation and numerical calculation of multiple integrals is important. This paper provides some techniques (i.e., binomial series and integration term by term theorem) to obtain the infinite series forms of two types of multiple integrals. In fact, the applications of the two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and employ Maple to solve these problems.

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