Calculating Some Integrals of Trigonometric Functions

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Abstract

In this article, we study three types of integrals related with the trigonometric functions. The infinite series forms of the three types of integrals can be obtained using binomial series and integration term by term theorem. On the other hand, we propose some examples to do calculation practically. The research methods adopted in this paper is to find solutions through manual calculations and verify the answers using Maple.

Key Words: integrals; trigonometric functions; infinite series forms; binomial series; integration term by term theorem; Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests.

In calculus and engineering mathematics, there are many methods to solve the integral problems, for example, change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. This paper studies the following three types of integrals of trigonometric functions, which are not easy to obtain their answers using the methods mentioned above.

$$\int \sin^r x \cos^s x dx, \tag{1}$$

$$\int \tan^p x \sec^q x dx, \tag{2}$$

$$\int \cot^a x \csc^b x dx, \tag{3}$$

where r, s, p, q, a, b, x are real numbers, and $0 < x < \pi/2$. The infinite series forms of the three types of integrals can be determined by using binomial series and integration term by term theorem; these are the major results of this article (i.e., Theorems 1, 2 and 3). Adams et al. [1], Nyblom [2], and Oster [3] provided some methods to solve the integral problems. On the other hand, Yu [4-31], Yu and Chen [32], and Yu and Sheu [33-35] used some techniques: complex power series method, integration term by term theorem, Parseval's theorem, area mean value theorem, and generalized Cauchy integral formula to solve some types of integrals. In this paper, some examples are proposed to demonstrate the manual calculations, and we verify the results using Maple.

2. Methods and Results

First, we introduce two important methods used in this paper, which can be found in ([36, p244]) and ([36, p269]) respectively.

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2.1 Binomial series: $(1+x)^{\beta} = \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!} x^k$,

where β, x are real numbers, $|x|<1$, and for any positive integer k, $(\beta)_k = \beta(\beta-1)\cdots(\beta-k+1)$, $(\beta)_0 = 1$.

2.2 Integration term by term theorem:

Assume that $\{g_n\}_{n=0}^{\infty}$ is a sequence of Lebesgue integrable functions defined on I. If $\sum_{n=0}^{\infty}\int_I |g_n|$ is convergent, then $\int_I \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty}\int_I g_n$.

In the following, we determine the infinite series form of the integral (1).

Theorem 1 *Let r, s, x be real numbers, $0 < x < \pi/2$, and C be a constant.*

Case 1. If r is not a negative odd integer, then

$$\int \sin^r x \cos^s x dx$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{s-1}{2}\right)_k (-1)^k}{k!(2k+r+1)} (\sin x)^{2k+r+1} + C. \quad (4)$$

Case 2. If r is a negative odd integer, then

$$\int \sin^r x \cos^s x dx$$

$$= \sum_{\substack{k=0 \\ k \neq \frac{-(r+1)}{2}}}^{\infty} \frac{\left(\frac{s-1}{2}\right)_k (-1)^k}{k!(2k+r+1)} (\sin x)^{2k+r+1} + \frac{\left(\frac{s-1}{2}\right)_{\frac{-(r+1)}{2}} (-1)^{\frac{-(r+1)}{2}}}{\left(\frac{-(r+1)}{2}\right)!} \ln(\sin x) + C. \quad (5)$$

Proof *Case 1.* If is not a negative odd integer, then

$$\int \sin^r x \cos^s x dx$$

$$= \int y^r (1-y^2)^{\frac{s-1}{2}} dy$$

(where $y = \sin x$)

$$= \int y^r \sum_{k=0}^{\infty} \frac{\left(\frac{s-1}{2}\right)_k}{k!} (-y^2)^k dy \quad (6)$$

(by binomial series)

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{s-1}{2}\right)_k (-1)^k}{k!(2k+r+1)} y^{2k+r+1} + C$$

(by integration term by term theorem)

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{s-1}{2}\right)_k (-1)^k}{k!(2k+r+1)} (\sin x)^{2k+r+1} + C.$$

Case 2. If r is a negative odd integer, then by Eq. (6) we have

$$\int \sin^r x \cos^s x dx$$

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$$= \sum_{\substack{k=0 \\ k \neq \frac{-(r+1)}{2}}}^{\infty} \frac{\left(\frac{s-1}{2}\right)_k (-1)^k}{k!(2k+r+1)} y^{2k+r+1}$$

$$+ \frac{\left(\frac{s-1}{2}\right)_{\frac{-(r+1)}{2}} (-1)^{\frac{-(r+1)}{2}}}{\left(\frac{-(r+1)}{2}\right)!} \ln y + C$$

$$= \sum_{\substack{k=0 \\ k \neq \frac{-(r+1)}{2}}}^{\infty} \frac{\left(\frac{s-1}{2}\right)_k (-1)^k}{k!(2k+r+1)} (\sin x)^{2k+r+1}$$

$$+ \frac{\left(\frac{s-1}{2}\right)_{\frac{-(r+1)}{2}} (-1)^{\frac{-(r+1)}{2}}}{\left(\frac{-(r+1)}{2}\right)!} \ln(\sin x) + C.$$

q.e.d.

Next, we obtain the infinite series forms of integrals (2) and (3).

Theorem 2 *Suppose that* p, q, x *are real numbers,* $0 < x < \pi/2$, *and* C *be a constant.*
Case 1. If p *is not a negative odd integer, then*

$$\int \tan^p x \sec^q x dx$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{-p-q-1}{2}\right)_k (-1)^k}{k!(2k+p+1)} (\sin x)^{2k+p+1} + C. \quad (7)$$

Case 2. If p *is a negative odd integer, then*

$$\int \tan^p x \sec^q x dx$$

$$= \sum_{\substack{k=0 \\ k \neq \frac{-(p+1)}{2}}}^{\infty} \frac{\left(\frac{-p-q-1}{2}\right)_k (-1)^k}{k!(2k+p+1)} (\sin x)^{2k+p+1}$$

$$+ \frac{\left(\frac{-p-q-1}{2}\right)_{\frac{-(p+1)}{2}} (-1)^{\frac{-(p+1)}{2}}}{\left(\frac{-(p+1)}{2}\right)!} \ln(\sin x) + C. \quad (8)$$

Proof Since $\int \tan^p x \sec^q x dx$

$$= \int \sin^p x \cos^{-p-q} x dx,$$

it follows from Theorem 1 that the desired results hold. q.e.d.

Theorem 3 *Let* a, b, x *be real numbers, and* $0 < x < \pi/2$.
Case 1. If $a+b$ *is not a positive odd integer, then*

$$\int \cot^a x \csc^b x dx$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{a-1}{2}\right)_k (-1)^k}{k!(2k-a-b+1)} (\sin x)^{2k-a-b+1} + C. \quad (9)$$

Case 2. If $a+b$ *is a positive odd integer, then*

$$\int \cot^a x \csc^b x dx$$

$$= \sum_{\substack{k=0 \\ k \neq \frac{a+b-1}{2}}}^{\infty} \frac{\left(\frac{a-1}{2}\right)_k (-1)^k}{k!(2k-a-b+1)} (\sin x)^{2k-a-b+1}$$

$$+ \frac{\left(\frac{a-1}{2}\right)_{\frac{a+b-1}{2}} (-1)^{\frac{a+b-1}{2}}}{\left(\frac{a+b-1}{2}\right)!} \ln(\sin x) + C. \quad (10)$$

Proof Since $\int \cot^a x \csc^b x dx$

$$= \int \sin^{-a-b} x \cos^a x dx,$$

by Theorem 1, we obtain the desired results.

q.e.d.

3. Examples

For the integral problems discussed in this article, three examples are proposed and we use Theorems 1, 2 and 3 to determine their infinite series forms. On the other hand, we employ Maple to calculate the approximations of some definite integrals and their solutions to verify our answers.

Example 1 Using Eq. (4) yields

$$\int \sin^{2/3} x \cos^{6/5} x dx$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{10}\right)_k (-1)^k}{k!(2k+5/3)} (\sin x)^{2k+5/3} + C, \tag{11}$$

for all $0 < x < \pi/2$.

Thus, the following definite integral

$$\int_{\pi/4}^{\pi/3} \sin^{2/3} x \cos^{6/5} x dx$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{10}\right)_k (-1)^k}{k!(2k+5/3)} \left[\left(\frac{\sqrt{3}}{2}\right)^{2k+5/3} - \left(\frac{\sqrt{2}}{2}\right)^{2k+5/3}\right]. \tag{12}$$

In the following, we use Maple to verify the correctness of Eq. (12).

>evalf(int((sin(x))^(2/3)*(cos(x))^(6/5),x=Pi/4..Pi/3),14);

0.12255210837130

>evalf(sum(product(1/10-j,j=0..(k-1))*(-1)^k/(k!*(2*k+5/3))*((sqrt(3)/2)^(2*k+5/3)-(sqrt(2)/2)^(2*k+5/3)),k=0..infinity),14);

0.12255210837129

Example 2 By Eq. (8), we obtain

$$\int \tan^{-3} x \sec^4 x dx$$

$$= \sum_{\substack{k=0 \\ k \neq 1}}^{\infty} \frac{(-1)_k (-1)^k}{k!(2k-2)} (\sin x)^{2k-2} + \ln(\sin x) + C. \tag{13}$$

for all $0 < x < \pi/2$.

Therefore,

$$\int_{\pi/6}^{\pi/4} \tan^{-3} x \sec^4 x dx$$

$$= \sum_{\substack{k=0 \\ k \neq 1}}^{\infty} \frac{(-1)_k (-1)^k}{k!(2k-2)} \left[\left(\frac{\sqrt{2}}{2}\right)^{2k-2} - \left(\frac{1}{2}\right)^{2k-2}\right] + \ln(\sqrt{2}). \tag{14}$$

We also employ Maple to verify the correctness of Eq. (14).

>evalf(int((tan(x))^(-3)*(sec(x))^4,x=Pi/6..Pi/4),14);

1.5493061443340

>evalf(1+ln(sqrt(2))+sum(product(-1-j,j=0..(k-1))*(-1)^k/(k!*(2*k-2))*((sqrt(2)/2)^(2*k-2)-(1/2)^(2*k-2)),k=2..infinity),14);

1.5493061443341

Example 3 By Eq. (10), we obtain

$$\int \cot^7 x \csc^{-2} x dx$$

$$= \sum_{\substack{k=0 \\ k \neq 2}}^{\infty} \frac{(3)_k (-1)^k}{k!(2k-4)} (\sin x)^{2k-4} + 3\ln(\sin x) + C, \tag{15}$$

for all $0 < x < \pi / 2$.

Hence,

$$\int_{\pi/6}^{\pi/3} \cot^7 x \csc^{-2} x dx$$

$$= \sum_{\substack{k=0 \\ k \neq 2}}^{\infty} \frac{(3)_k (-1)^k}{k!(2k-4)} \left[\left(\frac{\sqrt{3}}{2} \right)^{2k-4} - \left(\frac{1}{2} \right)^{2k-4} \right]$$

$$+ 3\ln(\sqrt{3}). \tag{16}$$

Using Maple to verify the correctness of Eq. (16) as follows:

>evalf(int((cot(x))^7*(csc(x))^(-2),x=Pi/6.. Pi/3),14);

0.95347398855776

>evalf(-4/9+3*ln(sqrt(3))+sum(product(3-j ,j=0..(k-1))*(-1)^k/(k!*(2*k-4))*((sqrt(3)/2) ^(2*k-4)-(1/2)^(2*k-4)),k=3..infinity),14);

0.95347398855776

4. Conclusion

In this paper, we use binomial series and integration term by term theorem to solve some integral problems related with the trigonometric functions. In fact, the applications of the two methods are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our results.

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