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Common Fixed Point of Mappings Satisfying E.A. and CLR_f Properties

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Abstract.

In this paper, first we prove common fixed point theorems for a pair of weakly compatible maps along with E.A. property. Secondly, we prove common fixed point theorems for weakly compatible mappings along with CLR_f property.

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1. Introduction

Fixed point theorems have extensive applications in proving the existence and uniqueness of the solutions of differential equations, integral equations, partial differential equations and in other related areas. Over since last 50 years, fixed point theory has been revealed itself as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in diverse fields such as in biology, chemistry, economics, engineering, game theory and physics.

The point at which the curve y = f (x) and the line y = x intersects gives the solution of the curve i.e. the point of intersection is the fixed point of the curve. The usefulness of the concrete applications has increased enormously due to the development of accurate techniques for computing fixed points.

The aim of this paper is to study common fixed point of weakly compatible mappings satisfying E.A. and CLR_f properties. The following definitions and results will be needed in the sequel.

Definition 1.1. Two self-mappings f and g of a metric space (X, d) are said to be weakly commuting if

(1.1) $d(fgx, gfx) \le d(gx, fx)$ for all x in X.

Further, Jungck [6] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity.

Definition 1.2. Two self-mappings f and g of a metric space (X, d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X. In 1996, Jungck [7] introduced the concept of weakly compatible maps as follows:

Definition1.3. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Definition 1.4. A mapping T : X → X, where (X,d) is a metric space, is said to be sequentially weakly contraction if

(1.2) $d(Tx, Ty) \le d(x, y) - f_n(d(x,y))$

(f_n:I (interval or subset of R) →R)

where $x,y \in X$ and $f_n(t)$ is a sequence of function which converges uniformly to t, and monotonic function such that $f_n(t) = 0$ if and only if $t = 0$.

In 2002, Aamriet. al. [1] introduced the notion of E.A. property as follows:

Definition 1.5. Two self-mappings f and g of a metric space (X, d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

Theorem 1.6. Let (X, d) be a metric space and let f and g be weakly compatible self-maps of X satisfying the followings:

(1.3) f and g satisfy the E.A. property,

(1.4) fX is closed subset of X.

(1.5) $\psi(d(gx,gy)) \le \psi(d(fx, fy)) - f_n(d(fx,fy))$,

(f_n: I(interval or subset of R)→R) for all $x,y \in X$

where ψ : $[0, \infty) \to [0, \infty)$ is mappings with $\psi(0)=0$, $f_n(t)>0$ also $f_n(t)$ is a uniformly convergent sequence which converges to ψ (t) and $\psi(t) > 0$ for all $t > 0$.

Suppose also that either

(1.6) ψ is continuous and $\lim_{n\to\infty} t_n = 0$, if $\lim_{n\to\infty} f_n(t_n) = 0$.

or

(1.7) ψ is monotonic non-decreasing and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} f_n(t_n) = 0$.

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x_0$ for some $x_0 \in X$. Now, fX is closed subset of X, therefore, for $z \in X$, we have $\lim_{n\to\infty} fx_n = fz$.

We claim that fz = gz.

From (1.5), we have

$\psi(d(gx_n, gz)) \leq \psi(d(fx_n, fz)) - f_n((d(fx_n, fz))$ for all $n \in \mathbb{N}$.

Letting $n \to \infty$, we have

$$\psi(d(fz, gz)) \leq \lim_{n\to\infty} \psi(d(fx_n, fz)) - \lim_{n\to\infty} f_n((d(fx_n, fz)) = \psi(d(fz, fz)) - f_n((d(fz, fz)) = \psi(0) - f_n((0).$$

If (1.6) holds, then

$\psi(d(fz, gz)) \leq 0$, implies that $d(fz, gz) = 0$, that is, fz = gz.

If (1.7) holds, then

$d(fz, gz) \leq 0$, implies that fz = gz.

Therefore, fz = gz.

Now, we show that gz is the common fixed point of f and g.

Suppose that gz ≠ ggz. Since f and g are weakly compatible i.e.,gfz = fgz and therefore ffz = ggz. [since fz = gz]

From (1.5), we have

$$\psi(d(gz, ggz)) \leq \psi(d(fz, fgz)) - f_n((d(fz, fgz)) = \psi(d(gz, gfz)) - f_n((d(gz,gfz)) \quad \text{[since fz = gz]}$$
$$= \psi(d(gz, ggz)) - f_n((d(gz, ggz)). \quad \text{[since fz = gz]}$$

If (1.6) holds, then

$\psi(d(gz, ggz)) < \psi(d(gz, ggz))$, a contradiction.

If (1.7) holds, then

$d(gz, ggz) < d(gz, ggz)$, a contradiction.

Hence ggz = gz. Hence gz is the common fixed point of f and g.

Uniqueness :

Let u and v be two common fixed points of f and g such that u ≠ v.

From (1.5), we have

$$\psi(d(u, v)) = \psi(d(gu, gv)) \leq \psi(d(fu, fv)) - f_n((d(fu, fv)) = \psi(d(u, v)) - f_n((d(u, v)).$$

If (1.6) holds, then we have

$\psi(d(u, v)) < \psi(d(u, v))$, a contradiction.

If (1.7) holds, then we have

$d(u, v) < d(u, v)$, a contradiction.

Therefore, u = v, which proves the uniqueness.

Theorem 1.7. Let (X, d) be a metric space and let f and g be weakly compatible self-maps of X satisfying (1.3), (1.4) and the following :

(1.8) $\psi(d(gx, gy)) \leq \psi(N(fx, fy)) - f_n(N(fx, fy))$,

where $N(fx, fy) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{d(fx,gy)+d(fy,gx)}{2}\}$,

for all $x,y \in X$, where $f_n(x)$ is sequence of function which converges uniformly to $\psi(x)$ and $f_n(0) = 0$ and $f_n(t) > 0$ for all $t > 0$ and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} f_n(t_n) = 0$ and ψ: $[0, \infty) \to [0, \infty)$ is a mapping with $\psi(0) = 0$ and ψ (t) > 0 for all t > 0.

Suppose also that either

(1.9) ψ is continuous

or

(1.10) ψ is monotone non-decreasing and for all k > 0, $f_n(k) > \psi(k^+) - \psi(k^-)$, where $\psi(k^-)$ is the left limit of ψ at k.

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x_0$ for some $x_0 \in X$. Now, fX is closed subset of X, therefore, for $z \in X$, we have $\lim_{n\to\infty} fx_n = fz$.

We claim that fz = gz. Suppose that fz ≠ gz.

From (1.8),we have

$\psi(d(gx_n, gz)) \leq \psi(N(fx_n, fz)) - f_n((N(fx_n, fz))$ for all $n \in \mathbb{N}$.

Letting $n \to \infty$, we have

$$\psi(d(fz, gz)) \leq \lim_{n\to\infty} \psi(N(fx_n, fz)) - \lim_{n\to\infty} f_n((N(fx_n, fz)) = \psi(d(fz, gz)) - f_n((d(fz, gz)),$$ since $\lim_{n\to\infty} N(fx_n, fz) = d(fz, gz)$.

If (1.9) holds, then we have

$\psi(d(fz, gz)) < \psi(d(fz, gz))$, a contradiction.

If (1.10) holds, then we have

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d(fz, gz) < d(fz, gz), a contradiction.
Therefore, fz = gz.
Now, we show that gz is the common fixed point of f and g.
Suppose that gz ≠ ggz. Since f and g are weakly compatible i.e., gfz = fgz and therefore ffz = ggz.
From (1.8), we have
ψ(d(gz, ggz)) ≤ ψ(N(fz, fgz)) – f_n((N(fz, fgz))
= ψ(N(gz, gfz)) –f_n((N(gz, gfz))
[since fz = gz]
= ψ(N(gz, ggz)) –f_n((N(gz, ggz))
=ψ(d(gz, ggz)) – f_n((d(gz, ggz)), since N(gz, ggz) = d(gz, ggz).
If (1.9) holds, then we have
ψ(d(gz, ggz)) < ψ(d(gz, ggz)), a contradiction.
If (1.10) holds, then we have
d(gz, ggz) < d(gz, ggz), a contradiction.
Hence ggz = gz. Hence gz is the common fixed point of f and g.

Uniqueness:

Let u and v be two common fixed points of f and g such that u ≠ v.
From (1.8) , we have
ψ(d(u, v)) = ψ(d(gu, gv))
≤ ψ(N(fu, fv)) – f_n((N(fu, fv))
= ψ(N(u, v)) – f_n((N(u, v))
= ψ(d(u, v)) – f_n((d(u, v)), .
since N(u, v) = d(u, v).
If (1.9) holds, then we have
ψ(d(u, v)) < ψ(d(u, v)), a contradiction.
If (1.10) holds, then we have
d(u, v) < d(u, v), a contradiction.
Therefore, u = v, which proves the uniqueness.

In 2011, Sintunavarat et. al. [12] introduced the notion of CLR_f property as follows:

Definition 1.8. Two self-mappings f and g of a metric space (X, d) are said to satisfy CLR_f property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ = fx for some x in X.

Theorem 1.9. Let (X, d) be a metric space and let f and g be weakly compatible self-mappings of X satisfying (1.5), (1.6), (1.7) and the following:
(1.11) f and g satisfy CLR_f property.
Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the CLR_f property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ = fx for some x ∈ X.
Now, we claim that fx = gx.
From (1.5), we have
ψ(d(gx_n, gx)) ≤ ψ(d(fx_n, fx)) –f_n((d(fx_n, fx)) for all n ∈ ℕ.
Letting n ⟶ ∞, we have
ψ(d(fx, gx)) ≤ $\lim_{n\to\infty}\psi$(d(fx_n, fx))
– $\lim_{n\to\infty} f_n$((d(fx_n, fx))
= ψ(d(fx, fx)) - f_n((d(fx, fx))
= ψ(0) - f_n(0).
If (1.6) holds, then we have
ψ(d(fx, gx)) ≤ 0, implies that d(fx, gx) = 0, that is, gx = fx.
If (1.7) holds, then we have
d(fx, gx) ≤ 0, that is , gx = fx.
Therefore fx = gx.
Let w = fx = gx.
Since f and g are weakly compatible i.e., gfx = fgx, implies that, fw = fgx = gfx = gw.
Now, we claim that gw = w. Let, if possible, gw ≠ w.
If (1.6) holds, then from (1.5), we have
ψ(d(gw, w)) =ψ(d(gw, gx))
≤ ψ(d(fw, fx)) – f_n((d(fw, fx))
< ψ(d(fw, fx))
=ψ(d(gw, w)), a contradiction.
If (1.7) holds, then we have
d(gw, w) < d(gw, w), a contradiction.
Thus, we get gw = w = fw.
Hence w is the common fixed point of f and g.
Uniqueness:
Let u be another common fixed point of f and g such that fu = u = gu.
Now, we claim that w = u.
Let, if possible, w ≠ u.
If (1.6) holds, then from (1.5), we have
ψ(d(w, u)) = ψ(d(gw, gu))
≤ ψ(d(fw, fu)) – f_n((d(fw, fu))
= ψ(d(w, u)) – f_n((d(w,u))
<ψ(d(w, u)), a contradiction.
If (1.7) holds, then we have
d(w, u) < d(w, u), a contradiction.
Thus, we get, w = u.
Hence w is the unique common fixed point of f and g.

Theorem 1.10. Let (X, d) be a metric space and let f and g beweakly compatible self-mappings of X satisfying (1.8), (1.9), (1.10) and (1.11), then f and g have a unique common fixed point.

Proof. Since f and g satisfy the CLR_f property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ = fx for some x ∈ X.
Now, we claim that fx = gx.
From (1.8), we have

ψ(d(gx$_n$, gx)) ≤ ψ(N(fx$_n$, fx)) – f$_n$((N(fx$_n$, fx)) for all n ∈ ℕ.

Letting n → ∞, we have

$\psi(d(fx, gx)) \leq \lim_{n\to\infty} \psi(N(fx_n, fx)) - \lim_{n\to\infty} f_n((N(fx_n, fx))$

$= \psi(d(fx, fx)) - f_n((d(fx, fx))$

$= \psi(0) - f_n(0)$,

since $\lim_{n\to\infty} N(fx_n, fx) = d(fx, fx) = 0$.

If (1.9) holds, then we have

ψ(d(fx, gx)) ≤ 0, implies that d(fx, gx) = 0, that is, fx = gx.

If (1.10) holds, then we have

d(fx, gx) ≤ 0, that is, fx = gx.

Thus, we get, gx = fx.

Let w = fx = gx.

Since f and g are weakly compatible gfx = fgx, implies that, fw = fgx = gfx = gw.

Now, we claim that gw = w.

Let, if possible, gw ≠ w.

From (1.8), we have

ψ(d(gw, w)) = ψ(d(gw, gx))

≤ ψ(N(fw, fx)) – f$_n$((N(fw, fx))

= ψ(d(fw, fx)) – f$_n$((d(fw, fx))

since N(fw, fx) = d(fw, fx).

= ψ(d(gw, w)) – f$_n$((d(gw, w))

If (1.9) holds, then we have

ψ(d(gw, w)) < ψ(d(gw, w)), a contradiction.

If (1.10) holds, then we have

d(gw, w) < d(gw, w), a contradiction.

Thus, we get gw = w = fw.

Hence w is the common fixed point of f and g.

Uniqueness :

Let u be another common fixed point of f and g such that fu = u = gu.

We claim that w = u. Let, if possible, w ≠ u.

From (1.8), we have

ψ(d(w, u)) = ψ(d(gw, gu))

≤ ψ(N(fw, fu)) –f$_n$((N(fw,fu))

= ψ(d(fw, fu)) – f$_n$((d(fw, fu))

since N(fw, fu) = d(fw, fu).

= ψ(d(w, u)) – f$_n$((d(w, u))

If (1.9) holds, then we have

ψ(d(w, u)) < ψ(d(w, u)), a contradiction.

If (1.10) holds, then we have

d(w, u) < d(w, u), a contradiction.

Thus, we get w = u.

Hence w is the unique common fixed point of f and g.

Example1.11. Let X = [0, 1] be endowed with the Euclidean metric d(x, y) = $|x-y|$ and x≠y also let gx = ax and fx = (a+1)x for each x ∈ X. Then

d(gx, gy) = $a|x-y|$

and d(fx, fy) = $(a+1)|x-y|$.

Let ψ(t) = ct and f$_n$(t) = cnt/(n+t). Then

ψ(d(gx, gy)) = $\psi(a|x-y|) = ca|x-y|$

ψ(d(fx, fy)) = $\psi((a+1)|x-y|)$

= c$(a+1)|x-y|$

f$_n$(d(fx, fy)) = f$_n$($(a+1)|x-y|$)

=n$(a+1)|x-y|$ /(n+$(a+1)|x-y|$).

Now

ψ(d(fx, fy)) −f$_n$(d(fx, fy)) =

c$(a+1)|x-y|$) [1−(n/(n+$(a+1)|x-y|$)]

Since [1−(n/(n+$(a+1)|x-y|$)] >0

So ψ(d(gx, gy)) < ψ(d(fx, fy)) - f$_n$(d(fx, fy)).

From here, we conclude that f, g satisfy the relation (1.5).

Consider the sequence {x$_n$} = $\{\frac{1}{n}\}$ so that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ = 0 = f(0), hence the pair (f, g) satisfy the CLR$_f$ property. Also f and g are weakly compatible and 0 is the unique common fixed point of f and g.

From here, we also deduce that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ = 0, where 0 ∈ X, implies that f and g satisfy E.A. property.

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