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Convergence of Infinite Series - A Review

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ABSTRACT: *This paper reviews the methods to select correct tests for checking the convergence or divergence of infinite series. To check the convergence or divergence of a series, there are various methods like Power Test, Comparison Test, Root Test, Ratio Test, Gauss Test etc. and the conditions and methods to apply these tests are explained here. Only the correct use of these tests gives us the valid results about the behavior of a series.*

KEY WORDS: Convergent, Divergent; Oscillates; artial Sums; Alternating Series; Harmonic Series; Geometric series; Power Series.

I. INTRODUCTION:

An infinite series is a sequence of numbers in which an infinite number of terms are added successively. It is a series in which the number of terms increases without bound. Of particular concern with infinite series is whether they are convergent or divergent. For example, the infinite series 1+1+1+1+-------. is clearly divergent because the sum of the first n terms increases without bound as more and more terms are taken. [9] It is less clear as to whether the harmonic and alternating harmonic series:

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ ----------

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$ ----------

Converge or diverge. Indeed one may be surprised to find that the first is divergent and the second is convergent. What we shall do here is to consider some simple convergence tests for infinite series and explain their applications under different conditions. [8]

1.1 SEQUENCE OF n^{th} PARTIAL SUMS:

Before studying the behavior of a series, we shall know about the sequence of $\boldsymbol{n^{th}}$ partial sums:

Let $\sum a_n$ be an infinite series with partial sums as

$\sigma_1 = a_1$

$\sigma_2 = a_1 + a_2$

$\sigma_3 = a_1 + a_2 + a_3$

Then $\{\sigma_n\}$ is called sequence of n^{th} partial sums.

1.2 CONVERGENCE OF A SERIES:

Let $\sum a_n$ be an infinite series and $\{\sigma_n\}$ be sequence of n^{th} partial sums of $\sum a_n$ then

(i) $\sum a_n$ is convergent iff $\{\sigma_n\}$ is convergent

Further $\sum a_n$ = l iff $\{\sigma_n\} \to$ l

i.e. Sum of series $\sum a_n$ = Limit of sequence $\{\sigma_n\}$

(ii) $\sum a_n$ Diverges to $\pm\infty$ iff $\{\sigma_n\}$ diverges to $\pm\infty$.

(iii) $\sum a_n$ Oscillates finitely (or infinitely) iff $\{\sigma_n\}$ oscillates finitely (or infinitely). [2]This is also called the behavior of infinite series. Also there are various tests which shows that while the series is convergent or divergent.

II. DIFFERENT TESTS FOR CHECKING CONVERGENCE OF AN INFINITE SERIES:

There are four types of series:

1. Alternating series
2. Power series
3. Geometric series
4. Positive term series

The following table shows different tests used for different types of series:

Table 1: Different tests for different types of series:

Sr. No.	Type of series	Name of Test
1.	Alternating series	Leibnitz Test
2.	Power series	Power test
3.	Geometric series	Geometric Series Test
4.	Positive term series	Comparison Test Cauchy's Condensation Test Cauchy's Root Test Cauchy's Integral Formula D'Alembert's Ratio Test

		Raabe's Test Gauss Test Logarithmic Test

2.1 LEIBNITZ TEST:

2.1.1 Applicability: It is also known as **alternating series test** because with the help of this test, we can check the convergence of an **Alternating series.** [1]

2.1.2 Conditions to Apply and Results:

If the sequence $\{v_n\}$ is

(i) Monotonic

(ii) $\{v_n\} \to 0$

Then $\sum(-1)^{n-1}\, v_n$ is convergent

EXAMPLE 1. : The series $\sum \frac{(-1)^{n-1}}{n}$ is convergent.

Here $v_n = \frac{1}{n}$

Clearly (i) $\{\frac{1}{n}\}$ is monotonically decreasing sequence.

(ii) $\frac{1}{n} \to 0$ as $n \to \infty$

So by Leibnitz Test, $\sum \frac{(-1)^{n-1}}{n}$ is convergent.

2.2 POWER TEST (P- TEST):

2.2.1 Applicability: 1. This test is only used when the given series is of the form $\sum \frac{1}{n^p}$ and it is useful in various Powerful tests like Comparison test, Caucy's Condensation Test etc. [5]

2.2.2 Conditions to Apply and Results: $\sum \frac{1}{n^p}$ is divergent for $p \le 1$

Convergent for $p > 1$

EXAMPLE 2: The series $\frac{1}{n^{1/2}}$ is divergent

Because here $p = \frac{1}{2} < 1$

2.3 GEOMETRIC SERIES TEST:

2.3.1 Applicability: This test is used only for Geometric series in which common ratio of two consecutive terms is equal. [3]

2.3.2 Conditions to Apply and Results:

Let the given series be of the form $1+x+x^2+x^3+$----

Then it converges for $|x| < 1$

And diverges for $x \ge 1$

2.4 COMPARISON TEST:

2.4.1 Applicability: 1. This test is used only when the given series can be broken in to two parts u_n and v_n and behavior of v_n can be easily checked most probably with the help of Power Test and Geometric Test.

2.4.2 Conditions to Apply and Results: Let $\sum u_n$ and $\sum v_n$ be two series and $\lim_{n\to\infty} \frac{u_n}{v_n} = l$

(i) If $l \neq 0$, l is finite then $\sum u_n$ and $\sum v_n$ behave alike.

(ii) If $l = 0$ then $\sum u_n$ is convergent if $\sum v_n$ is convergent.

(iii) If l = infinite then $\sum u_n$ is divergent if $\sum v_n$ is divergent.

EXAMPLE 3: The series $\sum \frac{n}{n^2+n+1}$ is divergent.

Here $u_n = \frac{n}{n^2+n+1}$

$\& v_n = \frac{1}{Degree\ of\ denominator - Degree\ of\ numerator} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n^2+n+1} \text{ x } n$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}+\frac{1}{n^2}} = 1 \neq 0$$

Hence $\sum u_n$ and $\sum v_n$ behave alike.

Now $\sum v_n = \sum \frac{1}{n}$ is divergent. (By P-Test)

Therefore $\sum u_n = \sum \frac{n}{n^2+n+1}$ is also divergent.

2.5 CAUCHY'S CONDENSATION TEST:

2.5.1 Applicability: This test is only used when we know the monotonic behavior of a series. If the series is monotonically decreasing, then apply Cauchy Condensation test. [6]

2.5.2 Conditions to Apply and Results: Let $\sum a_n$ be a series such that

(i) $a_n > 0 \quad \forall\, n$

(ii) $a_n > a_{n+1}$

Then $\sum a_n$ and $\sum 2^n\, a_{2^n}$ behave alike.

EXAMPLE 4: The series $\sum \frac{(\log n)^{-3/4}}{n}$ is divergent.

Here $\sum a_n = \sum \frac{(\log n)^{-3/4}}{n}$

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(i) Clearly $a_n = \frac{1}{n(\log n)^{3/4}} > 0 \quad \forall\ n$

(ii) $\frac{1}{(n+1)[\log(n+1)]^{3/4}} < \frac{1}{n(\log n)^{3/4}}$

$\therefore a_{n+1} < a_n$

Hence $\sum a_n$ and $\sum 2^n a_{2^n}$ behave alike.

$\therefore \sum 2^n a_{2^n} = \sum 2^n \frac{1}{2^n(\log 2^n)^{3/4}}$

$= \sum \frac{1}{(n \log 2)^{3/4}}$

$= \frac{1}{(\log 2)^{3/4}} \sum \frac{1}{n^{3/4}}$ which is divergent by P-Test (Here p=$\frac{3}{4}$ <1)

Hence $\sum \frac{(\log n)^{-3/4}}{n}$ is divergent.

2.6 *CAUCHY'S ROOT TEST:*

2.6.1 Applicability: This test is used when a_n is having powers like n, n^2 etc.

2.6.2 Conditions to Apply and Results: Let $\sum a_n$ be a series such that

(i) $a_n \geq 0 \quad \forall$ n

(ii) $\lim_{n\to\infty} a_n^{1/n} = l$

Then $\sum a_n$ is convergent if l < 1

Divergent if l > 1

If l = 1 then Cauchy's Root Test fails. [4]

EXAMPLE 5: The series $\sum(\frac{n}{n+1})^{n^2}$ is convergent

Here (i) $a_n = (\frac{n}{n+1})^{n^2} \geq 0 \quad \forall$ n

(ii) $\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} (\frac{n}{n+1})^{\frac{n^2}{n}}$

$= \lim_{n\to\infty}(\frac{1}{1+\frac{1}{n}})^n$

$= \frac{1}{e} < 1$

Hence $\sum(\frac{n}{n+1})^{n^2}$ is convergent.

2.7. *CAUCHY'S INTEGRAL FORMULA:*

2.7.1 Applicability: 1. When limits are given i.e. we have to prove that the given series lies between two limits then Cauchy's Integral Test is applied.

2. When limits are not given between which we have to prove that the given series lies but $\int f(x)\,dx$ is very easy to calculate then we apply Cauchy's Integral Test.

2.7.2 Conditions to Apply and Results: Let f be defined as non-negative and decreasing $\forall$ n$\geq$ 1.

Let (i) σ_n= f (1) +f (2) +f (3) +----------+f (n)

(ii) And $I_n = \int_1^n f(x)\,dx$

Then $\{\sigma_n - I_n\}$ is convergent and $\sum f(n)$ *and* $\lim_{n\to\infty} \int_1^n f(x)\,dx$ behave alike.

And if $\sum f(n)$ is convergent then it converges between I and I + f (1), where I= $\lim_{n\to\infty} I_n$

2.8 *D'ALEMBERT'S RATIO TEST:*

2.8.1 Applicability: If a_n contains factorial of any term, then we apply Ratio Test.

2.8.2 Conditions to Apply and Results: Let $\sum a_n$ be a series such that

(i) a_n>0 $\forall$ n

(ii) $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = l$

Then $\sum a_n$ is convergent if l > 1

Divergent if l < 1

If l = 1 then Ratio Test fails.

EXAMPLE 6: The series $\sum \frac{(n!)^2}{2n!} x^n$; x > 0 is convergent for x < 4 and divergent for x > 4.

Here (i) $a_n = \frac{(n!)^2}{2n!} x^n > 0 \quad \forall$ n

(ii) $a_{n+1} = \frac{(n+1!)^2}{2(n+1)!} x^{n+1}$

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$

$= \lim_{n\to\infty} \frac{(n!)^2(2n+2)(2n+1)(2n)!}{(2n)!(n+1)^2\,(n!)^2} \frac{1}{x}$

$= \lim_{n\to\infty} \frac{4n(1+\frac{1}{2n})}{n(1+\frac{1}{n})} \frac{\mathbf{1}}{\boldsymbol{x}}$

$= \frac{\mathbf{4}}{\boldsymbol{x}}$

Hence $\sum a_n$ is convergent if $\frac{4}{x} > 1$

Divergent if $\frac{4}{x} < 1$

i.e. $\sum a_n$ is convergent if x < 4

Divergent if x > 4

And for x=4, Ratio Test fails.

2.9 *RAABE'S TEST:*

2.9.1 Applicability: 1. When we apply Ratio Test but it fails and conditions to apply the most common tests i.e. Gauss Test and Logarithmic Test are not satisfied then we apply Raabe'sTest.

2.9.2 Conditions to Apply and Results: Let $\sum a_n$ be a series such that

(i) a_n>0 $\forall$ n

(ii) $\lim_{n\to\infty}(\frac{a_n}{a_{n+1}} - 1)n = l$

Then $\sum a_n$ is convergent if l > 1
Divergent if l < 1

EXAMPLE 7: In example 6, if x=4 then Ratio Test fails

Now $\lim_{n\to\infty}(\frac{a_n}{a_{n+1}} - 1)n = \lim_{n\to\infty}(\frac{1+\frac{1}{2n}}{1+\frac{1}{n}} - 1)\, n$

$= \lim_{n\to\infty}(\frac{\frac{2n+1}{2n}}{\frac{n+1}{n}} - 1)n$

$= \lim_{n\to\infty}(\frac{2n+1}{2(n+1)} -1)\, n$

$= \lim_{n\to\infty}(\frac{2n+1-2n-2}{2(n+1)})n$

$= \lim_{n\to\infty}(\frac{-n}{2(n+1)})$

$= -\frac{1}{2} < 1$

$\therefore \sum a_n$ is divergent.

2.10 *GAUSS TEST OR μ TEST:*

2.10.1 Applicability: 1. If in Ratio Test, it is possible to expand $\frac{a_n}{a_{n+1}}$ in powers of $\frac{1}{n}$ then we apply Gauss Test.

2.10.2 Conditions to Apply and Results: Let $\sum a_n$ be a series such that

(i) $a_n > 0 \quad \forall$ n

(ii) $\frac{a_n}{a_{n+1}} = 1 + \mu(\frac{1}{n}) + O(\frac{1}{n^2})$

Then $\sum a_n$ is convergent if $\mu > 1$
Divergent if $\mu \le 1$

EXAMPLE 8: In example 6, if x=4 then Ratio Test fails.

Here $\frac{a_n}{a_{n+1}} = \frac{1+\frac{1}{2n}}{1+\frac{1}{n}}$

$= (1+\frac{1}{2n})\,(1+\frac{1}{n})^{-1}$

$= (1+\frac{1}{2n})[1-\frac{1}{n} + O(\frac{1}{n^2})]$

$= 1+\frac{1}{n}(\frac{1}{2} - 1) + O(\frac{1}{n^2})$

Here $\mu = -\frac{1}{2} < 1$

$\therefore \sum a_n$ is divergent.

2.11 *LOGARITHMIC TEST:*

2.11.1 Applicability: 1. If **in** Ratio Test, $\frac{a_n}{a_{n+1}}$ involves exponential function, then apply Logarithmic Test. [7]

2.11.2 Conditions to Apply and Results: Let $\sum a_n$ be a series such that

(i) $a_n > 0 \quad \forall$ n

(ii) $\lim_{n\to\infty} n \log(\frac{a_n}{a_{n+1}}) = l$

Then $\sum a_n$ is convergent if l > 1
Divergent if l < 1

EXAMPLE 9: The series $\sum \frac{n^n x^n}{n!}$ is divergent for $x \ge \frac{1}{e}$

Here $a_n = \frac{n^n x^n}{n!}$ and $a_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

$\frac{a_n}{a_{n+1}} = \frac{n^n x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1} x^{n+1}}$

$= \frac{1}{(1+\frac{1}{n})^n x} \to \frac{1}{ex}$

$\therefore$ By Ratio test, $\sum a_n$ is convergent if $\frac{1}{ex} > 1$ i.e. $x < \frac{1}{e}$
Divergent if $\frac{1}{ex} < 1$ i.e. $x > \frac{1}{e}$

For $x = \frac{1}{e}$ Ratio Test fails

$\therefore$ For $x = \frac{1}{e}$, $\frac{a_n}{a_{n+1}} = \frac{e}{(1+\frac{1}{n})^n}$

$\lim_{n\to\infty} n \log(\frac{a_n}{a_{n+1}}) = \lim_{n\to\infty} n \log(\frac{e}{(1+\frac{1}{n})^n})$

$= \lim_{n\to\infty} n\,[\log e - n \log(1+\frac{1}{n})]$

$= \lim_{n\to\infty} n[1- n\,(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \text{-----})]$

$= \lim_{n\to\infty} \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \text{--------}$

$= \frac{1}{2} < 1$

Hence the series $\sum \frac{n^n x^n}{n!}$ is divergent for $x \ge \frac{1}{e}$

TEST	CONDITIONS	OUTCOME
Leibnitz Test (For Alternating Series)	The given series is of the form $\sum(-1)^{n-1} v_n$ and the sequence $\{v_n\}$ is (i) monotonic (ii) $\{v_n\} \to 0$	$\sum(-1)^{n-1} v_n$ is convergent
Power test (For Power Series)	Given series is of the form $\sum \frac{1}{n^p}$	Convergent for p > 1 And divergent for p≤1
Geometric Series Test (For Geometric Series)	The given series be of the form $1+x+x^2+x^3+$---	converges for |x| <1 And diverges for x ≥ 1
Comparison Test (For Rational Terms)	$\sum u_n$ and $\sum v_n$ are two series & $\lim_{n\to\infty} \frac{u_n}{v_n} = l$	(i) If l ≠ 0, l is finite then $\sum u_n$ and $\sum v_n$ behave alike. (ii) If l = 0 then $\sum u_n$ is convergent if $\sum v_n$ is convergent. (iii) If l = infinite then $\sum u_n$ is divergent if

		$\sum v_n$ is divergent.
Cauchy's Condensation Test (For monotonically decreasing series)	$\sum a_n$ is a series such that (i) $a_n>0\ \forall$ n (ii) $a_n > a_{n+1}$	$\sum a_n$ and $\sum 2^n\, a_{2^n}$ behave alike.
Cauchy's Root Test (For a series having powers like n, n^2 etc.)	$\sum a_n$ is a series such that (i) $a_n \geq 0\ \forall$ n (ii) $\lim_{n\to\infty} a_n{}^{1/n}$ = l	$\sum a_n$ is convergent if l < 1 Divergent if l > 1
Cauchy's Integral Formula (when $\int f(x)$ is easy to calculate)	If f be defined as non-negative and decreasing ∀ n≥ 1. And (i) σ_n = f(1)+f(2)+f(3)+----------+f(n) (ii) $I_n = \int_1^n f(x)\,dx$	{ $\sigma_n - I_n$} is convergent & $\sum f(n)$ & $\lim_{n\to\infty}\int_1^n f(x)\,dx$ behave alike. And $\sum f(n)$ converges between I &I + f (1), where I= $\lim_{n\to\infty} I_n$
D'Alembert's Ratio Test (For a series having factorials and combination of factorials and powers)	$\sum a_n$ is a series such that (i) $a_n>0\ \forall$ n (ii) $\lim_{n\to\infty}\frac{a_n}{a_{n+1}}$ = l	$\sum a_n$ is convergent if l > 1 Divergent if l < 1 If l = 1 then Ratio Test fails.
Raabe's Test (In case when Ratio test fails and Gauss and Logarithmic test are also not satisfied)	$\sum a_n$ is a series such that (i) $a_n>0\ \forall$ n (ii) $\lim_{n\to\infty}(\frac{a_n}{a_{n+1}}-1)n$ = l	$\sum a_n$ is convergent if l > 1 Divergent if l < 1
Gauss Test (When Ratio Test Fails and It is possible to expand $\frac{a_n}{a_{n+1}}$ in powers of $\frac{1}{n}$	$\sum a_n$ is a series such that $a_n>0\ \forall$ n (ii) $\frac{a_n}{a_{n+1}}$ = 1+ $\mu(\frac{1}{n})$ + O($\frac{1}{n^2}$)	$\sum a_n$ is convergent if μ > 1 Divergent if $\mu \leq$ 1
Logarithmic Test (When Ratio Test Fails and $\frac{a_n}{a_{n+1}}$ involves exponential functions)	$\sum a_n$ is a series such that $a_n>0\ \forall$ n (ii) $\lim_{n\to\infty} n\log(\frac{a_n}{a_{n+1}})$ = l	$\sum a_n$ is convergent if l > 1 Divergent if l < 1

3. Conclusion:

With the correct use of above discussed tests, valid results can be found about the behavior of a series. So precaution should be taken while selecting the tests of convergence for different types of series. Otherwise one gets invalid results. That is why selection of a correct series convergence test is much important. Moreover, there is no universal criterion for verification of the behavior of series.

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