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Generalized Differential Transformation Method for Solving System of Linear Volterra Integro-Differential Equations of Fractional Order

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Abstract

In this paper, the Generalized Differential Transformation Method (GDTM) for approximating the solution of systems of linear volterra integro-differential equations of fractional of fractional is implemented. The fractional derivative is considered in the Caputo sense. The approximate solutions are calculated in the form of a convergent series with easily quantifiable workings. Numerical results show that this approach is easy to implement and accurate when applied to systems integro-differential equations.

Keywords: Integro-Differential equations, Fractional calculus, Generalized Differential,Transformation method , system of linear Volterra integro-differential equations .

This paper is prearranged as: Section one shows some basic concept that we will need. section two GDTM Technique for solving linear integro-differential equations of fractional order . modified GDTM Technique for solving systems of linear integro-differential equations of fractional order has been dispraised in third section. illustrated example solved in three cases with table of results and diagrams in section four . Conclusions is proposed in section five.

1. Basic concept

1.1 Fractional Calculus

The history of Fractional Calculus is as old as that of the classical calculus. Based on L'Hopital and Leibniz (1695),Fractional Calculus.The Fractional Calculus is a theory of arbitrary real or even complex order. It

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is a generalization of the classical calculus and therefore conserves many of the fundamental properties. As an intensively rising area of the calculus during the last couple decades it offers wonderful new feature for research and thus becomes more and more in use in various applications[4].There is some basic definitions in fractional calculus mention the most important.

Definition (1.1): (Caputo Fractional Derivatives D_C^{α}),[4],[10],[20]:

Let $f(t) \in C_{\mu}^{n}$ [18] that is defined on the closed interval [a,b], the Caputo fractional derivative of order $\alpha > 0$ of f is defined by:

$$D_c^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau\,, & n-1 < \alpha < n\,, \quad n \in N \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n, \quad n \in N \end{cases} \qquad (1.1)$$

Definition (1.2): (Riemann-Liouville Fractional Intrgrals),[16],[18]:

Let $f(t) \in C_{\mu}^{n}$ that is defined on the closed interval [a,b], the Riemann-Liouville Fractional integral of order $\alpha > 0$ of f is defined by: $J^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau$ (1.2)

Definition(1.3)(Gamma function),[12], [14],[20]

The complete gamma function $\Gamma(t)$ is also known as generalized factorial function. It is defined by using the following integral:

$$\Gamma(t) = \int_0^{\infty} S^{t-1} e^{-S} dS\,, \quad t > 0 \quad , S \text{ any variable} \qquad (1.3)$$

(1.5)(Properties of Gamma function), [14],[20]:

(1) $\Gamma(t+1) = t\Gamma(t) \quad t > 0$

(2) $\Gamma(t) = (t-1)!$ **t is positive integer, convention: 0! = 1**

(3) $\Gamma\frac{1}{2} = \sqrt{\pi}$

(1.2) Differential transform DT

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There is many numerical methods that have been adopted to resolve this type of problems such as Adomian decomposition method (ADM), variational iteration method (VIM), homotopy analysis method(HAM), homotopy perturbation method (HPM) and differential transform method(DTM),[13],[15]. Generalized Differential transform (GDT) has taken the shape of an important and convenient tool. In(1980) G.E. Pukhov used differential transform in numerical methods to solve fractional differential equations for the first time[8],[9].The factual using of DTM was in (1986) by Zhou in electric circuit analysis,[24]. Since then, (DTM) was success-fully applied for a large variety of problems. In (2008) Erturk and Momani proposed (DTM) as efficiency tool to solve systems of fractional differential equations [5],[11].After this many researchers used (DTM) till Taghvafard and Erjaee in (2011) solved systems of singular Volterra integro-differential equations of convolution type with DT [6].

So the differential transform method can be defined as a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation [2],[3],[8],[19].

Fractional Differential transform can be defined as:

$$F(k)=\begin{cases} \frac{k}{\square} \in Z^+, \frac{1}{\left(\frac{K}{\square}\right)!}\left[\frac{d^{\frac{k}{\square}} f(x)}{dx^{k/\square}}\right]_{x=x_0} \textbf{for} \quad k=0,1,\dots,(q\alpha-1) \\ \frac{k}{\square} \notin Z^+, 0 \end{cases} \quad (1.4)$$

Where α is the order of fractional derivative[5],[7]: .

And We define the generalized differential transform of the kth derivative of function f(t) in one variable as follows,[6]:: $F(k)=\frac{1}{\Gamma(\alpha k+1)}\left[\left(D_{t_\circ}^{\alpha}\right)^{k} f(t)\right]_{t=t_\circ}$ **(1.5)**

where $(D_{t_\circ}^{\alpha})^{K}=D_{t_\circ}^{\alpha}.D_{t_\circ}^{\alpha}\dots\dots D_{t_\circ}^{\alpha}$**,k-times and the differential inverse transform of** $F(k)$**is defined as follows:** $f(t)=\sum_{k=0}^{\infty}F_{\alpha}(k)\left(t-t_\circ\right)^{\alpha k}$ **(1.6)**

(1.6)(Properties of GDTM),[5],[6],[7],[22]

1-If $f(t) = g(t) \pm h(t)$, **then** $F(k) = G(k) \pm H(k)$.

2-If f(t) =ag(t), then $F(k) = aG(k)$, **where a is a constant.**

3-If f(t) =g(t)h(t), then $F(k) = \sum_{l=0}^{k} G(l)\, H(k-l)$

4-If $f(t)=g_1(t)g_2(t), \ldots\ldots, g_{n-1}(t)g_n(t)$, **then** $f(x)=\sum_{k_{n-1}=0}^{k} \sum_{k_{n-1}=0}^{k_{n-1}} \ldots\ldots \sum_{k_{n-1}=0}^{k} \sum_{k_{n-1}=0}^{k} G_1(k_1)G_n(k_2 - k_1)\ldots\ldots G_{n-1}(k_{n-1} - k_{n-2})G_n(k - k_{n-1})$

5-If f(t)= $D_{t_\circ}^{\alpha} g(t)$, then $F(k) = \frac{\Gamma(\alpha k+1)+1}{\Gamma(\alpha k+1)} G(k+1)$

6-If $f(t)=(t - t_\circ)^{\beta}$, **then** $F(k) = \delta\left(k - \frac{\beta}{\alpha}\right)$, **where** $\delta(k) = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ if } k \neq 0 \end{cases}$

7-If $f(t)=\int_{t_\circ}^{t} g(t)\, dt$,**then** $F(k) = \frac{G\left(k-\frac{1}{\alpha}\right)}{\alpha k}$ **where** $k \geq \frac{1}{\alpha}$

8-If $f(t)=g(t)\int_{t_\circ}^{t} h(t)\, dt$ **then** $F(k) = \sum_{k_1=\frac{1}{\alpha}}^{k} \frac{H\left(k-\frac{1}{\alpha}\right)}{\alpha k_1} G(k - k_1)$ **where** $k \geq \frac{1}{\alpha}$

9-If $f(t)=\int_{t_\circ}^{t} h_1(t)h_2(t) \ldots\ldots\ldots h_{n-1}(t)h_n(t)\, dt$, **then**

$$F(k) = \frac{1}{\alpha k} \sum_{k_{n-1}=0}^{k-\frac{1}{\alpha}} \sum_{k_{n-2}}^{k_{n-1}} \ldots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} H_1(k_1)H_2(k_2 - k_1)..H_{n-1}\left(k_{n-1} - k_n - \frac{1}{\alpha}\right], k \geq \frac{1}{\alpha}.$$

10- If $f(t)=[g_1(t)g_2(t) \ldots\ldots\ldots\ldots g_{m-1}(t)g_m(t)] \int_{t_\circ}^{t} h_1(t)h_2(t) \ldots . h_{n-1}(t)h_n(t)dt$,

then $F(k) = \sum_{k_i}^{k} \frac{1}{\alpha k_1} \sum_{j_{n-1}=0}^{k_1-\frac{1}{\alpha}} \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \sum_{j_1=0}^{k-k_1} \sum_{i_{m-2}=0}^{i_{m-1}} G_1(i_1)G_2(i_2 - i_1) \ldots G_{m-1}(i_{m-1} - i_{m-2})G_m(k - i_{m-1} - k_1) \times H_1(j_1)H_2(j_2 - j_1) \ldots\ldots H_{n-1}(j_{n-1} - j_{n-2})H_n(k_1 - j_{n-1} - \frac{1}{\alpha})$, where $k \geq 1/\alpha$.

2. Solving Linear Volterra Integro-Differential Equations of Fractional order(L-FVIDE) using GDTM Technique [5],[6]:

A few searchers involve with fractional systems ,so the propose technique provide a good results for linear system of fractional integro-differential equations. First we will give a technique that used to solve fractional linear integro-differential equations ,which is based on Taylor series expansion. Consider the L-FVIDE $D_c^{\beta} u(t) = f(t) + \lambda \int_0^t K(t,x)u(x)\, dx$ (2.1)

With initial condition u(0)= a , $0 < \beta \leq 1$, $\lambda \in \mathbb{R}$, $D_c^{\beta} u(t)$ **denotes the Caputo fractional derivative of order** β **for u(t),f(t) is continues function with** $f(t) \in C_{\mu}^{n}, t \in [a,b]$.**To solve the equation** (2.1) **using GDTM, one can take the**

differential transform to the both sides of equation (2.1). According to GDTM properties in (1.6),the terms of equation (2.1) can be transform as following:

1-$D_c^{\beta}\, u(t)$ transformed to $\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)} U(k+\frac{\beta}{\alpha})$

2-$f(t)$ transformed to $F(k)$

3-$\lambda \int_0^t K(t,x)u\ (x)dx$ transformed to $\frac{1}{\alpha k} U\left(k-\frac{1}{\alpha}\right) \lambda\ F\{K(t,x)\}$,

In this part of the transform, k satisfies that k $\geq \frac{1}{\alpha}$,taking into consideration what is suitable for each function in terms of transformation. Next one can characterize the new equation to find U(k+$\frac{\beta}{\alpha}$), $k = 0, \ldots, n$. such that

$$U(k+\frac{\beta}{\alpha}) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)} [\ F(k) + \frac{1}{\alpha k} U\left(k-\frac{1}{\alpha}\right) \lambda\ F\{K(t,x)\}\ ] \qquad (2.2)$$

Now we have two cases:

First case When $\beta = \alpha = 1$

To transform the initial condition of (2.1) we need to use the following relation at t=a

$$U(k_\circ) = \begin{cases} \text{If } \alpha k \in \mathbb{Z}^+ \text{ then } U(k_\circ) = \frac{1}{\alpha k_\circ}\frac{du}{dt} \\ \text{If } \alpha k \notin \mathbb{Z}^+ \text{ then } U(k_\circ) = 0 \quad \forall\ k_\circ = 0, \ldots, n \end{cases}$$

where $k_\circ = \frac{\beta}{\alpha} - 1$, at t=0 .

It is clear that $k_\circ = 0$ in this case, and by substituting $\beta = \alpha = 1$ the value of U(k+$\frac{\beta}{\alpha}$) will be U(k+1). Next substituting k values in the obtained equation $\forall\ k = 0, \ldots, n$.

One can find the values of U(k+1) $\forall\ k = 0, \ldots, n$ which present the transformed series of U(k+1) ,after this depending on the derivations of equation (1.6) in section (1.6).Taking the inverse transform of equation (2.2) by using the following relation

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_\circ)^{\alpha k} \qquad t_\circ = 0\ , \alpha = 1$$

$u(t) = \sum_{k=0}^{\infty} U(k)t^{\alpha k}$

We get the semi analytic solution for equation (2.1) in series form.

Second case When β is fractional

In this case selecting α **must satisfie:**

- $\alpha \leq \beta - 1$.
- $\frac{\beta}{\alpha} \in \mathbb{Z}^{+}$

By the same way we can substitute values of β and α **in U(** $k+\frac{\beta}{\alpha}$), $k = 0, \dots, n$.

and apply the same steps to obtain the transformed initial condition .Then, take k values $\forall k = 0, \dots, n$, **to find U(** $k+\frac{\beta}{\alpha}$) $\forall k = 0, \dots, n$. **after this, take the inverse transform of equation (2.2) :**$u(t) = \sum_{k=0}^{\infty} U(k)(t - t_{\circ})^{\alpha k}$ $\quad t_{\circ} = 0$ **,** α is fractional

$$u(t) = \sum_{k=0}^{\infty} U(k)t^{\alpha k}$$

to obtain the approximate solution for the original equation (2.1) in series form.

3. Solving system of Linear Volterra Integro-Differential Equations of Fractional order Using GDTM Technique

We can generalize the technique of GDTM obtained in section two above [5],[6] to solve a system of L-FVIDE as fallowing:

Consider the system of L-FVIDE $\sum_{j=1}^{n} D_c^{\beta} \; u_j(t) = f_j(t) + \lambda_j \int_0^t K_j(t, x, u_j) u_j(x) dx \qquad (3.1)$

With initial conditions $u_j(0) = a$, $0 < \beta \leq 1$, j=1,2,..,m , i=1,2,..,n $m, n, \in \mathbb{Z}^{+}$, $\lambda_j \in \mathbb{R}$

$D_c^{\beta} u_j(t)$ denotes the caputo fractional derivative of order β for $u_j(t), f_j(t)$ iscontinuous function with $f(t) \in C_{\mu}^{n}, t \in [a, b]$ .To solve the system (3.1) by using GDTM, one can take the differential transform for both sides of system(3.1).According to GDTM properties in (1.6),the terms of equation (3.1)can be transformed as following

1-$D_c^{\beta} u_j(t)$ transformed to $\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)} U_j(k+\frac{\beta}{\alpha})$

2-$f_j(t)$transformed to $F_j(k)$

3-$\lambda_j \int_0^t K_j(t, x) u_j \; (x) dx$ transformed to $\frac{1}{\alpha k} U_j\left(k - \frac{1}{\alpha}\right) \lambda_j \; F\{K_j(t, x)\}$, In this part of the transform, k satisfies $k \geq \frac{1}{\alpha}$,taking into consideration what is suitable for each function in terms of transformation.Next we can characterize the new equation to find $U_j(k+\frac{\beta_j}{\alpha})$, $j = 1,2,..,m$, $k = 0,\dots,n$ and $m,n \in \mathbb{Z}^+$. such that

$$U_j(k+\frac{\beta_j}{\alpha}) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta_j+1)}[F_j(k) + \frac{1}{\alpha k}U_j\left(k-\frac{1}{\alpha}\right)\lambda_j F\{K_j(t,x)\}] \qquad (3.2)$$

Now we have three cases:

First case When $\beta_j = \alpha = 1$

To transform the initial conditions we need to use the relation below

$$\text{at } t=a \quad U_j(k_\circ) = \begin{cases} \text{If } \alpha k \in \mathbb{Z}^+ \text{ then } U_j(k_\circ) = \frac{1}{\alpha k_\circ}\frac{du}{dt} \\ \text{If } \alpha k \notin \mathbb{Z}^+ \text{ then } U_j(k_\circ) = 0 \quad \forall k_\circ = 0,\dots,n, \forall j = 1,2,..,m \end{cases}$$

where $k_\circ = \frac{\beta_j}{\alpha} - 1$, a=0 . It is Clear that $k_\circ = 0$ in this case, and by substituting $\beta_j = \alpha = 1$ the value of $U_j(k+\frac{\beta_j}{\alpha})$ will be $U_j(k+1)$ $\forall$ j=1,2,..,m. Next substituting k values in the obtained equations $\forall k = 0,\dots,n$ .one can find the values of $U_j(k+1)$ $\forall k = 0,\dots,n$,which present the transformed series of $U_j(k+1)$, after this taking the inverse transform of equation (3.2) by using the relation $u_j(t)=\sum_{k=1}^{\infty} U_j(k)(t-t_0)^{\alpha k}$ $t_0=0$, $\alpha = 1$, $u_j(t) = \sum_{k=1}^{\infty} U_j(k)t^k$

we get the semi analytic solution for equation (3.1) in series frame.

Second case When β_j is fractional and $\beta_1 = \beta_2 = \beta_3 = \cdots = \beta_j$, $\forall j = 1,2,..,m$

selecting α must satisfie :

- $\alpha \leq \beta_j - 1$.
- $\frac{\beta_j}{\alpha} \in \mathbb{Z}^+$

using the same way as in section tow we can substitute values of β_j and α in $U_j(k+\frac{\beta_j}{\alpha}), k = 0,\dots,n$, and apply the same steps in section tow to obtain the transformed initial conditions $U_j(k_\circ)$.Then, we take k values $\forall k = 0,\dots,n$ to find $U_j(k_i+\frac{\beta_j}{\alpha}), \forall j = 1,2,..,m$, $\forall i = 0,\dots,n$. ,after this, we take the inverse transform of equation(3.2)

$u_j(t)=\sum_{k=1}^{\infty} U_j(k)(t-t_0)^{\alpha k}$ $t_0=0$, α is fractional

$u_j(t) = \sum_{k=0}^{\infty} U_j(k)t^{\alpha k}$

To obtain the approximation solution for the original equation (3.1) in series form.

Third case:-When β_j is fractional and $\beta_1 \neq \beta_2 \neq \beta_3 \neq \cdots \neq \beta_j$, $\forall j = 1,2,..,m$

Selecting α must satisfies:

- $\alpha \leq \beta_j - 1$.
- $\frac{\beta_j}{\alpha} \in \mathbb{Z}^+$

using the same way in section tow, one can substitute values of β_j and α in $U_j\ (k+\frac{\beta_j}{\alpha})$, $k = 0, \ldots, n$,and apply the same steps to obtain the transformed initial conditions $U_j(k_\circ)$.We may find out that the numbers of the transformed initial conditions $U_j(k_\circ)$are different for equation to other according to value of β_j for each equation. Then next take k values $\forall\, k = 0, \ldots, n$, to find $U_j\ (\ k_i+\frac{\beta_j}{\alpha}), \forall\ j = 1,2,\ldots, m\ \ ,\forall\ i = 0, \ldots, n$. after this, we take the inverse transform of equation (3.2)

$u_j\ (t)=\sum_{k=1}^{\infty} U_j\,(k)\,(t-t_0)^{\alpha k}$ $t_0=0$ α is fractional

$$u_j(t) = \sum_{k=0}^{\infty} U_j(k)\, t^{\alpha k}$$

in order to obtain the approximation solution for the original equation (3.1) in series form.

Next to illustrate the solution procedure and show the feasibility and efficiency of the GDTM we have applied the modified method to solve system of linear fraction Volterra integro-differential equations with known exact solution, and solve them in different cases to obtained different series.

4. Application example

Consider that we have a system of L-FVIDE $D_c^{\beta_1} u_1(t) = t - u_1(t) + \int_0^t [u_2(x) - u_1(x)]dx$

$$D_c^{\beta_2} u_2(t) = 3t + 3 - u_2(t) + \int_0^t [u_2(x) - u_1(x)]dx \qquad (4.1)$$

with initial conditions $u_1(0)=0$, $u_2(0)= 1$

the exact solution of system (4.1) is given in [1] as: $u_1\ (t)= t^2$, $u_2(t)= (1+t)^2$

To solve the system (4.1) using GDTM technique and the properties in (1.6) ,we get transformed the system below: For the first equation:

$$U_1\left(k+\frac{\beta}{\alpha}\right) = \frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)}\left[\delta\left(k-\frac{1}{\alpha}\right) - U_1(k) + \frac{1}{\alpha k}\left(U_2\left(k-\frac{1}{\alpha}\right) - U_1\left(k-\frac{1}{\alpha}\right)\right)\right] \qquad (4.1.a)$$

By the same way we can transform the second equation:

$$U_2\left(k+\frac{\beta}{\alpha}\right) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)}\left[3\delta\left(k-\frac{1}{\alpha}\right) - 3\delta(k) - U_2(k) + \frac{1}{\alpha k}\left(U_2\left(k-\frac{1}{\alpha}\right) - U_1\left(k-\frac{1}{\alpha}\right)\right)\right] \qquad (4.1.b)$$

First case:

To obtain the first solution: put $\alpha = \beta = 1$ which means $k_\circ$ =0 then $U_1\ (0) = 0$, $U_2\ (0) = 1$

Then substituting α and β values in equations (4.1.a), (4.1.b) we get:

$$U_1(k+1) = \frac{\Gamma(k+1)}{\Gamma(k+2)}\left[\delta(k-1) - U_1(k) + \frac{1}{k}\left(U_2(k-1) - U_1(k-1)\right)\right]$$

$$U_2(k+1) = \frac{\Gamma(k+1)}{\Gamma(k+2)}\left[3\delta(k-1) + 3\delta(k) - U_2(k) + \frac{1}{k}\left(U_2(k-1) - U_1(k-1)\right)\right] \qquad (4.1.c)$$

Substituting the values k in system equations (4.1.c), $\forall\, k = 0,1,2, \ldots$

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For k = 0 then :$U_1(1) = 0$,$U_2(1) = 2$.

For k = 1 then :$U_1(2) = 1$,$U_2(2) = 1$

By the same way we can obtain that: $U_1(k) = U_2(k) = 0 \;\forall\; k \geq 2$.

Now to get semi analytic solution for system (1.4) formed in a series form applying the inverse transform of system (4.1.c):

$u_1(t)=\sum_{k=1}^{\infty} U_1(k)(t-t_0)^{\alpha k}$ t_0=0, $\alpha = 1$, $u_1(t) = \sum_{k=0}^{\infty} U_1(k)t^k$

$u_1(t) = U_1(0)t^0 + U_1(1)t^1 + U_1(2)t^2 + U_1(3)t^3 \ldots\ldots\ldots$

Then $u_1(t)= t^2$ and this is exact solution for $u_1(t)$.

By the way we can find $u_2(t)$: $u_2(t) = \sum_{k=0}^{\infty} U_2(k)(t-t_0)^{\alpha k}$ t_0=0, $\alpha = 1.$,$u_2(t) = \sum_{k=0}^{\infty} U_2(k)t^k$

$u_2(t) = U_2(0)t^0 + U_2(1)t^1 + U_2(2)t^2 + U_3(3)t^3$

$u_2(t)= 1 + 2t + t^2 + 0 + \ldots\ldots$

Then $u_2(t)= (1+t)^2$ and this is exact solution for $u_2(t)$

Second case: For this case one can select the value of $\beta_j \;\forall\; j = 1,2$ as:

$\beta_1 = \beta_2$=0.5 and $\alpha = 0.5$ which means $k_\circ$ = 0 then U1(0) = 0 , U2(0) = 1

By substituting β_j and α values in equations (4.1.a), (4.1.b) we get:

$$U_1(k+1) = \frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}+1\right)}\left[\delta(k-2) - U_1(k) + \frac{2}{k}\left(U_2(k-2) - U_1(k-2)\right)\right]$$

$$U_2(k+1) = \frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}+1\right)}\left[3\delta(k-2) + 3\delta(k) - U_2(k) + \frac{2}{k}\left(U_2(k-2) - U_1(k-2)\right)\right] \quad (4.1.d)$$

Once again, substituting k values in system (4.1.d) $\forall\; k = 0,1,2,\ldots$

For k = 0 then :$U_1(1) = 0, U_2(1) = 2.2567583$

For k = 1 then :$U_1(2) = 0$,$U_2(2) = -0.2275461$

Continue this process for $\forall\; k \geq 2$ one can find $U_1(k+1)$, $U_2(k+1)$

Again applying the inverse transform of system (4.1.d) to get approximate solution for system (4.1) formed in a series form:

$u_1(t)=\sum_{k=1}^{\infty} U_1(k)(t-t_0)^{\alpha k}$ t_0=0, $\alpha = \frac{1}{2}$,$u_1(t) = \sum_{k=0}^{\infty} U_1(k)t^{\frac{1}{2}k}$

$u_1(t) = U_1(0)t^0 + U_1(1)t^{\frac{1}{2}} + U_1(2)t^1 + U_1(3)t^{\frac{3}{2}} + \cdots + \ldots . U(n)t^{\frac{n}{2}} + \cdots$ the $u_1(t)$= 0 + 0 + 0+ $(1.5045056)t^{\frac{3}{2}}$ +........

By the way we can find $u_2(t)$:$u_2(t) = \sum_{k=0}^{\infty} U_2(k)(t-t_0)^{\alpha k}$ t_0=0, $\alpha = \frac{1}{2}$. ,$u_2(t) = \sum_{k=0}^{\infty} U_2(k)t^{\frac{1}{2}k}$

$u_2(t) = U_2(0)\, t^0 + U_2(1)t^{\frac{1}{2}} + U_2(2)\, t + U_2(3)\, t^{\frac{3}{2}} + U_2(4)\, t^2 + U_2(5)\, t^{\frac{5}{2}} + \ldots U_2(n)\, t^{\frac{n}{2}} + \cdots$

$u_2(t) = 1 + (2.2567583)t^{\frac{1}{2}} + (-0.2275461)\, t + (3.1801833)\, t^{\frac{3}{2}} + \ldots$

Finding the arbitrary value of t for any fractional divertive of order β and calculate it:

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For $\beta = 0.8$ then $\alpha = 0.2$, By substituting β and α values in equations (4.1.a) ,(4.1.b) we get

$$U_1(k+4) = \frac{\Gamma\left(\frac{k}{5}+1\right)}{\Gamma\left(\frac{k}{5}+\frac{4}{5}+1\right)}\left[\delta(k-5) - U_1(k) + \frac{5}{k}\left(U_2(k-5) - U_1(k-5)\right)\right]$$

$$U_2(k+4) = \frac{\Gamma\left(\frac{k}{5}+1\right)}{\Gamma\left(\frac{k}{5}+\frac{4}{5}+1\right)}\left[3\delta(k-5) - 3\delta(k) - U_2(k) + \frac{5}{k}\left(U_2(k-5) - U_1(k-5)\right)\right] \quad (4.1.g)$$

For example take k=0 then: $U_1(4) = 0$. and by the same way we can find U_1 (5), U_1 (6),…, U_1 (10) $\forall k \geq 1$, applying the inverse transform to of system (4.1.g) get approximate solution for system (4.1) formed in series form:u_1 (t)=$\sum_{k=1}^{\infty} U_1(k)(t-t_0)^{\alpha k}$ t_0=0, $\alpha = \frac{1}{5}$

$$u_1(t) = U_1(0)t^0 + U_1(1)t^{\frac{1}{5}} + U_1(2)t^1 + U_1(3)t^{\frac{3}{5}} + \cdots + \ldots . U_1(n)\, t^{\frac{n}{5}} + \cdots$$

$u_1(t)$=0+….+ U_1 (9) $t^{\frac{9}{5}}$+ $\ldots . + U_1(n)\, t^{\frac{n}{5}} + \cdots$

$u_1(t)$= (1.1929681) $t^{\frac{9}{5}}$+ $\ldots . U_1(n)\, t^{\frac{n}{5}} + \cdots$

And one can find $u_2(t)$ for k=0 $U_2(4) = 2.1473425$

and by the same way we can find U_2 (5), U_2 (6),…, U_2 (10) $\forall k \geq 1$,

applying the inverse transform of system (4.1.g) to get approximate solution for system(4.1) formed in a series form:u_2 (t)=$\sum_{k=1}^{\infty} U_2(k)(t-t_0)^{\alpha k}$ t_0=0, $\alpha = \frac{1}{5}$

$$u_2(t) = U_2(0)t^0 + U_2(1)t^{\frac{1}{5}} + U_2(2)t^1 + U_2(3)t^{\frac{3}{5}} + U_2(4)t^{\frac{4}{5}} + \cdots + \ldots . U_2(n)\, t^{\frac{n}{5}} + \cdots$$

u_2 (t) = 1+0+… + $(2.1473425\,)t^{\frac{4}{5}} + \cdots + \ldots . U_2(n)\, t^{\frac{n}{5}} + \cdots$

u_2 (t) = 1+ $(2.1473425\,)t^{\frac{4}{5}}$+…. +$U_2(n)\, t^{\frac{n}{5}} + \cdots$

For t=0.6 , $\beta = 0.8$,$\alpha = 0.2$,one can obtain u_1 (t)= 0.47566487 , u_2 (t) = 6.06053573

The illustrated value of u(t) colored as red in table (2) below ,one can find the other entries values of table (2)by the same way.

Third case: Putting $D_c^{\beta_1} u_1(x) = x - u_1(x) + \int_0^x [u_2(t) - u_1(t)]dx$

$$D_c^{\beta_2} u_2(x) = 3x + 3 - u_2(x) + \int_0^x [u_2(t) - u_1(t)]dx$$

With initial conditions u_1(0)=0 , u_2(0)= 1 , the exact solution is given in [1] as:

u_1 (t)= t^2 , u_2(t)= $(1+t)^2$

$$U_1\left(k+\frac{\beta_1}{\alpha}\right) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta_1+1)}\left[\delta\left(k-\frac{1}{\alpha}\right) - U_1(k) + \frac{1}{\alpha k}\left(U_2\left(k-\frac{1}{\alpha}\right) - U_1\left(k-\frac{1}{\alpha}\right)\right)\right]$$

$$U_2\left(k+\frac{\beta_2}{\alpha}\right) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta_2+1)}\left[3\delta\left(k-\frac{1}{\alpha}\right) - 3\delta(k) - U_2(k) + \frac{1}{\alpha k}\left(U_2\left(k-\frac{1}{\alpha}\right) - U_1\left(k-\frac{1}{\alpha}\right)\right)\right] \quad (4.1.f)$$

put:- $\beta_1 = 0.5 = \frac{1}{2}$, $\beta_2 = 0.75 = \frac{3}{4}$ $\alpha = 0.25 = \frac{1}{4}$

To find transform initial conditions: $k_1 = \frac{\beta_1}{\alpha} - 1 = 1$ which means $k_1 = 0{,}1$

$k_2 = \frac{\beta_2}{\alpha} - 1 = 2$ which means $k_2 = 0,1,2$

Then: $U_1(0) = U_1(1) = 0$, $U_2(0) = 1$,$U_2(1) = U_2(2) = 0$

Then substituting α and β values in system (4.1. f) we get:

$$U_1(k+2) = \frac{\Gamma(\frac{k}{4}+1)}{\Gamma(\frac{k}{4}+\frac{1}{2}+1)}\left[\delta(k-4) - U_1(k) + \frac{4}{k}\left(U_2(k-4) - U_1(k-4)\right)\right]$$

$$U_2(k+3) = \frac{\Gamma\left(\frac{k}{4}+1\right)}{\Gamma\left(\frac{k}{4}+\frac{3}{4}+1\right)}\left[3\delta(k-4) - 3\delta(k) - U_2(k) + \frac{4}{k}\left(U_2(k-4) - U_1(k-4)\right)\right] \quad (4.1.h)$$

Substituting k values in system (4.1. h)$\forall k = 0,1,2, \ldots v$,For $k = 0$ then :$U_1(2) = 0$

By the same way one can obtain that $U_1(k+2) = 0$ For $k = 1,2,3$

For $k = 4$ then :$U_1(6) = 1.5045056$

and one can find $U_1(k+2)$ For $k = 5,6, \ldots.$ in similar way

To find U_2 , for $k = 0$ then:$U_2(3) = 2.1761305$

For $k = 1$ then :$U_2(4) = 0$. and by the same way we can find $U_2(k+3)$ $\forall k \geq 2$:

again applying the inverse transform of system (4.1.h) to get the approximate solution for system (4.1) formed in a series form:$u_1(t) = \sum_{k=1}^{\infty} U_1(k)(t-t_0)^{\alpha k}$ $t_0=0$, $\alpha = \frac{1}{4}$, $u(t) = \sum_{k=0}^{\infty} U(k) t^{\frac{1}{4}k}$

$u_1(t) = (1.5045056)t^{\frac{3}{2}} + (-t^2) + \cdots..$

By the way we can find $u_2(t)$: $u_2(t) = \sum_{k=1}^{\infty} U_2(k)(t-t_0)^{\alpha k}$ $t_0=0$, $\alpha = \frac{1}{4}$,$u_2(t) = \sum_{k=0}^{\infty} U_2(k) t^{\frac{1}{4}k}$

$u_2(t) = 1 + (2.1761305)t^{\frac{3}{4}} + (-1.5045056)t^{\frac{3}{2}} + \ldots.$

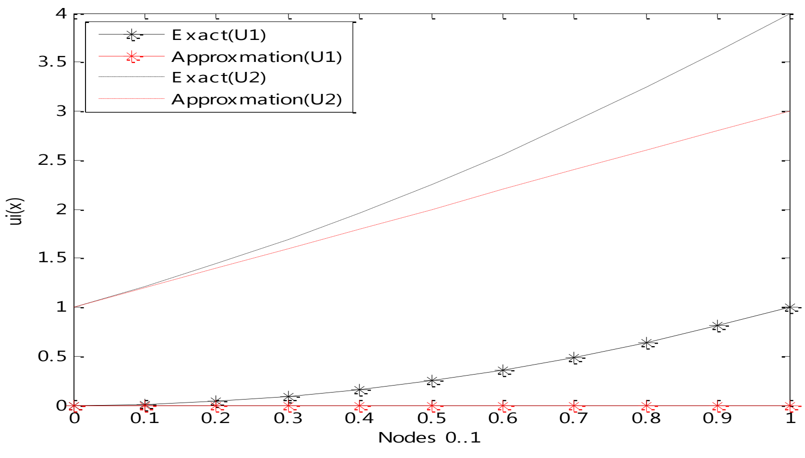

Figure (1) :Comparison between the exact solution and approximate solution of the example using GDTM when N=1 ,β=1

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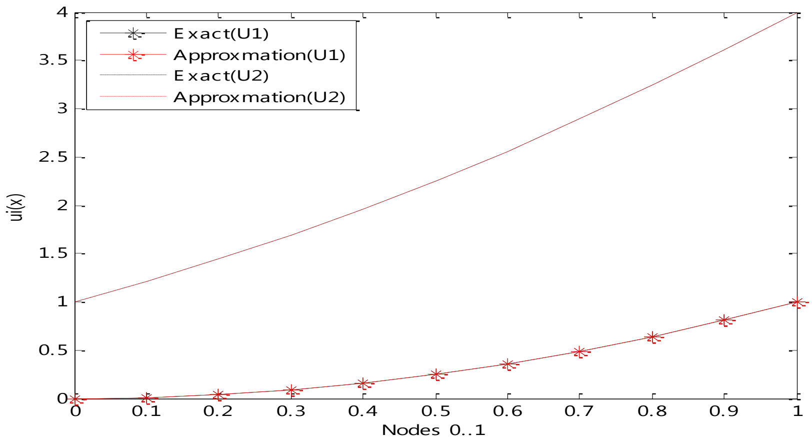

Figure (2):Comparison between the exact solution and approximate solution of the example using GDTM when N=10, β=1

Table(1) shows Results of system of the Linaer Volterra integro- differential equations of fractional order solved using GDTM after considering different values for N, $\beta = 1$

Values (	$\alpha = \beta = 1$ N=1		$\alpha = \beta = 1$ N=10	
	u_1	u_2	u_1	u_2
0.0	0.000000	1.000000	0.000000	1.0000
0.1	0.010000	1.20000(	0.010000	1.21000
0.2	0.040000	1.40000(	0.040000	1.44000
0.3	0.0900000	1.60000(	0.090000	1.69000
0.4	0.160000	1.800000	0.160000	1.96000
0.5	0.250000	2.00000(	0.250000	2.25000
0.6	0.360000	2.20000(	0.360000	2.56000
0.7	0.490000	2.40000(	0.49000	2.89000
0.8	0.640000	2.600000	0.640000	3.24000
0.9	0.810000	2.80000(	0.810000	3.61000
1.0	1.000000	3.00000(	1.000000	4.00000
	0.	3850		1.3323e-01

ans =	0	0.2303	(	4.4822e-0

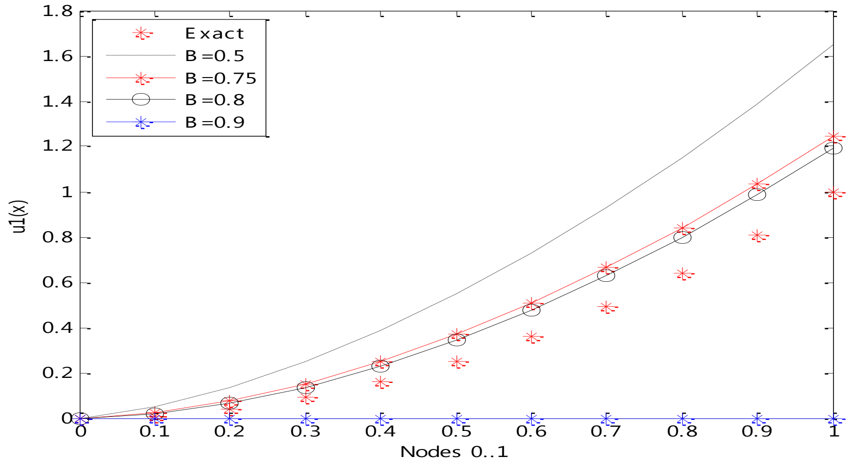

Figure (3):Comparison between the approximate solutions for u_1 in the example using GDTM when N=10 for different values for the fractional order divertive β

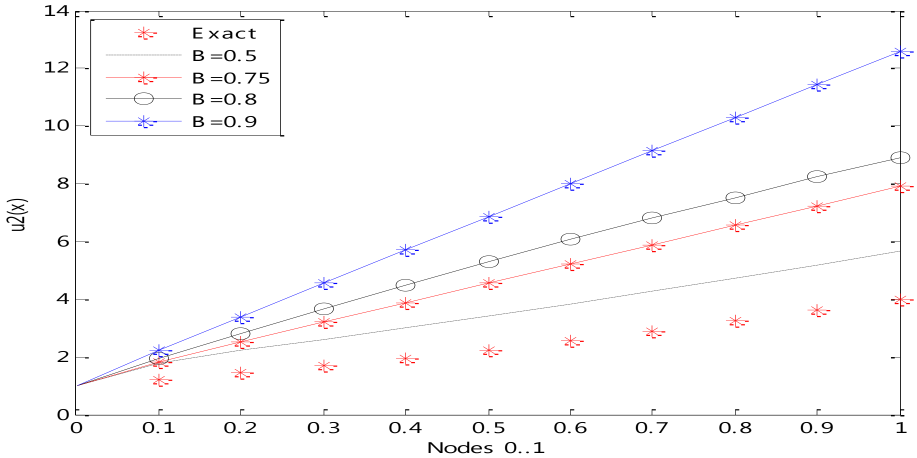

Figure (4):Comparison between the approximate solutions for u_2 in the example using GDTM when N=10 for different values for the fractional order divertive β

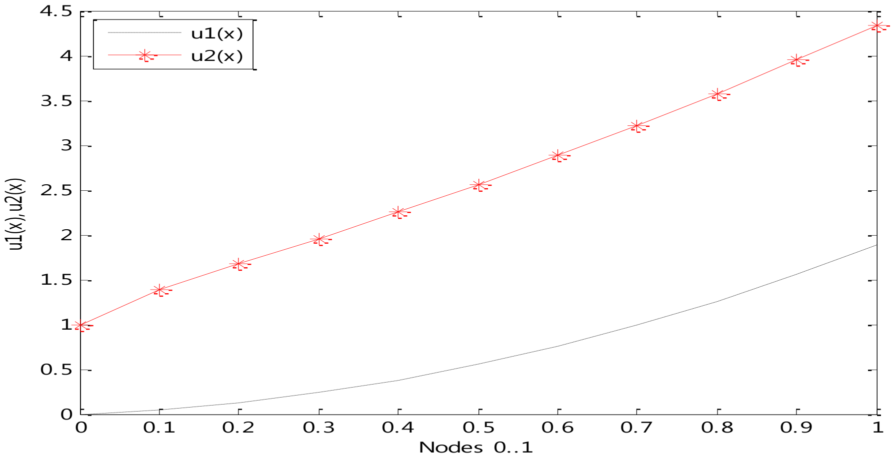

Figure (5)Comparison of the approximation solutions for u_1 and u_2 in the example using GDTM when N=10 for mixed values for the fractional order divertive β

Table (2)shows Results of system of Linaer Volterra integro- differential equations of fractional order solved using GDTM after considering different values for the fractional divertive β

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Values of t	$\beta = 0.5$		$\beta = 0.8$		$\beta = 0.9$	
	u_1	u_2	u_1	u_2	u_1	u_2
0.0	0.00000000	1.00000000	0.00000000	1.00000000	0.00000000	1.00000000
0.1	0.04766622	1.78221387	0.01890727	1.94280671	0.00000000	2.21314535
0.2	0.13554149	2.21419280	0.06583894	2.82017076	0.00000000	3.39122750
0.3	0.25072257	2.61887962	0.13659886	3.66842476	0.00000000	4.55772352
0.4	0.38909134	3.02102585	0.22926450	4.49073523	0.00000000	5.71702476
0.5	0.54862398	3.42912610	0.34259012	5.28800393	0.00000000	6.87113474
0.6	0.72832005	3.84725427	0.47566487	6.06053573	0.00000000	8.02120152
0.7	0.92786257	4.27789866	0.62777685	6.80842413	0.00000000	9.16796780
0.8	1.14749798	4.72296248	0.79834535	7.53167682	0.00000000	10.31195236
0.9	1.38800473	5.18421229	0.98688226	8.23026470	0.00000000	11.45353744
1.0	1.65070430	5.66352023	1.19296808	8.90414330	0.00000000	12.59301597

Table (3)showsResults of system of LinaerVolterra integro- differential equations of fractional order solved using GDTM after considering mixed values for the fractional order divertive β

Values of t	$\beta_1 = 0.5,\ \beta_2 = 0.75,\ \alpha = 0.25$ N=10	
	u_1	u_2
0.0	0.00000000	1.00000000
0.1	0.04389151	1.38613482
0.2	0.12631864	1.67522626
0.3	0.23913771	1.95993128
0.4	0.38133902	2.25321139
0.5	0.55323736	2.55994839
0.6	0.75562100	2.88234760
0.7	0.98947950	3.22143253
0.8	1.25589180	3.57761857
0.9	1.55597354	3.95097697
1.0	1.89085011	4.34137135

4. Conclusions

This present analysis exhibits the applicability of the differential transform method to solve systems of integro-differential equations of fractional order. The work emphasized our belief that the method is a reliable technique to handle linear fractional integro-differential equations. It provides the solutions in terms of convergent series with easily able to be gauged workings in a direct way without using linearization, perturbation or restrictive assumptions. The results of this method are in good agreement with the exact solution. in this method we do not need to do the difficult computation. The proposed method is talented and applicable to a broad class of linear and nonlinear problems in the theory of fractional calculus.

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