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A Study On Haar Integral and Its Methods

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ABSTRACT:

In this paper, an efficient numerical method for the solution of nonlinear partial differential equations based on the Haar wavelets approach is proposed. Approximate solutions of the generalized haar equation are compared with exact solutions. The proposed scheme can be used in a wide class of nonlinear reaction–diffusion equations. These calculations demonstrate that the accuracy of the Haar wavelet solutions is quite high even in the case of a small number of grid points. The present method is a very reliable, simple, small computation costs, flexible, and convenient alternative method. The study also compares the haar and wavelet transformation equations.

INTRODUCTION:

The past decade has witnessed the development of *wavelet analysis*, a new tool that emerged from mathematics and was quickly adopted by diverse fields of science and engineering. In the brief period since its creation in 1987-88, it has reached a certain level of maturity as a well-defined mathematical discipline, with its own conferences, journals, research monographs, and textbooks proliferating at a rapid rate. Wavelet analysis has begun to play a serious role in a broad range of applications, including signal processing, data and image compression, solution of partial differential equations, modelling multiscale phenomena, and statistics. There seem to be no limits to the subjects where it may have utility.

In this chapter we shall explore some additional topics that extend the basic ideas of wavelet analysis. We described the theory of wavelet packet transforms, which sometimes provide superior performance beyond that provided by wavelet transforms. A wavelet packet transform is a simple generalization of a wavelet transform. In this section I discussed the definition of wavelet transforms, and in the next section examine some examples illustrating their applications. All wavelet packet transforms are calculated in a similar way. Therefore we shall concentrate initially on the Haar wavelet packet transform, which is the easiest to describe. The Haar wavelet packet transform is usually referred to as the Walsh transform. [2]

Haar System:

The Haar orthogonal system begins with Φ (t),the characteristic function of the unit interval

$$\Phi(t) = x_{[0,1)}(t). \tag{1.1}$$

It is clear that $\Phi(t)$ and $\Phi(t - n)$, $n \neq 0$, $n \in Z$ are orthogonal since their product is zero. It is also clear that $\{\phi(t - n)\}$ is not a complete orthogonal system in $L^2(R)$ since its closed linear span V_o consists of 2 piecewise constant functions with possible jumps only at the integers. The characteristic function of (0,1/2), for example, with a jump at 1/2, can not have a convergent expansion.

In order to include more functions we consider the dilated version of ϕ (t) as well, ϕ (2^m t) where $m \in Z$. Then by a change of variable we see that $\{2^{m/2} \phi(2^m t - n)\}$ is an orthonormal system. V_m will denote its closed linear span. Since any function in L^2 (R) may be approximated by a piecewise constant function f_m with jumps at binary rationals, it follows that $\bigcup_n V_m$ is dense in L^2 (R). Thus the system $\{\phi_{mn}\}$ where

$$\phi_{mn}(t) = 2^{m/2} \ \Phi(2^m t - n), \tag{1.2}$$

is complete in L^2 (R), but, since ϕ (t) and ϕ (2t) are not orthogonal, it is not an orthogonal system. We must modify it somehow to convert it into an orthogonal system.

Fortunately the cure is simple; we let $\Psi(t) = \phi(2t) - \phi(2t - 1)$. Then everything works; $\{\psi(t - n)\}$ is orthonormal system, and $\psi(2t - k)$ and $\psi(t - n)$ are orthogonal for all k and n. This enables us to deduce that $\{\psi_{mn}\}_{m,n\in Z}$, where

$$\psi_{mn}(t) = 2^{m/2} \psi(2^m t - n), \tag{1.3}$$

is a complete orthonormal system in L^2 (R). this is the Haar system; the expansion of f $\in L^2$ (R)is

$$f(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle f, \psi_{mn} \rangle \psi_{mn}(t), \tag{1.4}$$

with convergence in the sense of L^2 (R). the standard approximation is the series given by

$$f_m(t) = \sum_{k=-\infty}^{m-1} \sum_{n=-\infty}^{\infty} \langle f, \psi_{kn} \rangle \psi_{kn}(t). \tag{1.5}$$

The ϕ (t) is usually called the scaling function in wavelet terminology while $\Psi(t)$ is the mother wavelet. [5]

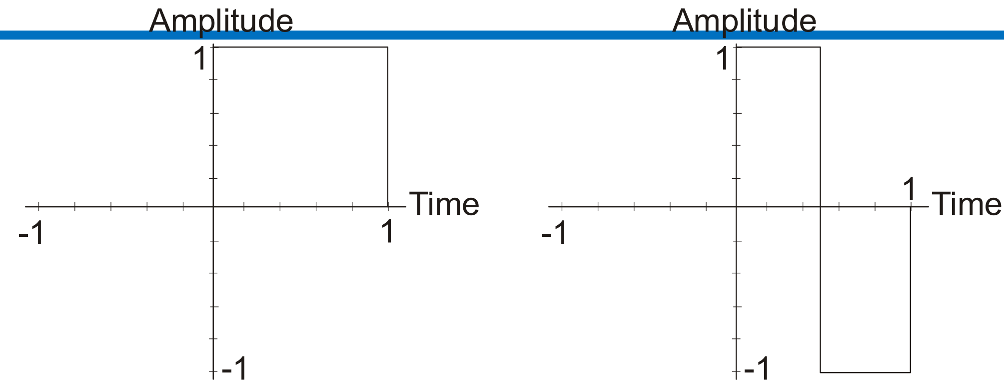


FIGURE 1: (a) The scaling function and (b) mother wavelet for the Haar system.

The Haar Transform to get that Wavelet feel

- Suppose for simplicity we assume an input vector x_k with $0 < k < 7$. This is readily decomposed into an obvious basis set as shown.

$$(x_k) = x_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ldots + x_7 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Other basis systems are of course possible (remember your QM and spinors?). In 1910 Haar proposed the following decomposition.

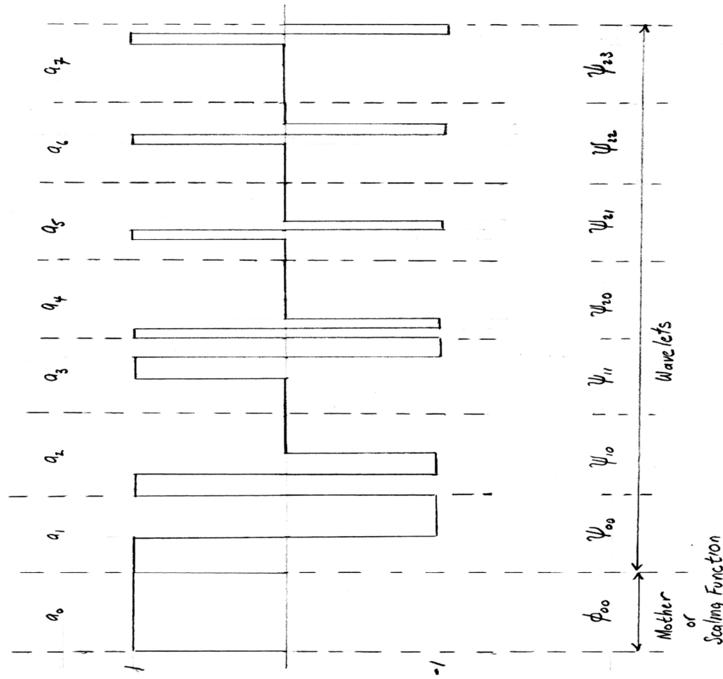
$$(x_k) = a_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + a_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + a_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + a_7 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

or $x_n = H_{nk} a_k$ with the columns of H being simply the above basis vectors and the a_k obtained by matrix inversion of H.

- These basis vectors have characteristic “shapes” when drawn on their side as shown in the figure on the next page and it is these shapes which show the essential features of what DWT decomposition does.

Notice:

1) A mother or scaling function at the start with a non-zero average. This will normally be normalised to 1.
2) Wavelet functions with zero average which are both compressed and translated. It is this compression and translation which finds peaks or pulses well.
3) The wavelet functions are orthogonal. You can see this directly by multiplying any two together.
4) The wavelet functions have compact support which means they are all localised. This is unlike the FT in which the basis functions $\exp(-2\pi nk / N)$ are continuous.



- How do we use other shapes and make a wavelet basis system out of them?

step 1: Mother functions

Let $\phi(x)$ be some mother function. The $\phi(2x)$ is the same function compressed by a factor of 2. Binary compression can therefore be denoted as $\phi_j = \phi(2^j x)$. Likewise $\phi(2x-1)$ is our compressed function translated by 1. Multiple translation and compression of the mother function can therefore be denoted as $\phi_{jk} = \phi(2^j x - k)$

We do ***not*** choose $\phi(x)$ arbitrarily but impose two conditions.

1) $\phi(x) = \sum_k c_k \phi(2x-k)$ or more generally $\phi(2^{j-1}x) = \sum_k c_k^j \phi(2^j x - k)$. That is it lends itself to a fractal like summing behaviour.

2) $\int \phi(x)dx = 1$, the normalisation condition. This leads to $\sum_k c_k = 2$. Alas, life is not easy and there is much confusion in the literature at this point. If you accept this as is

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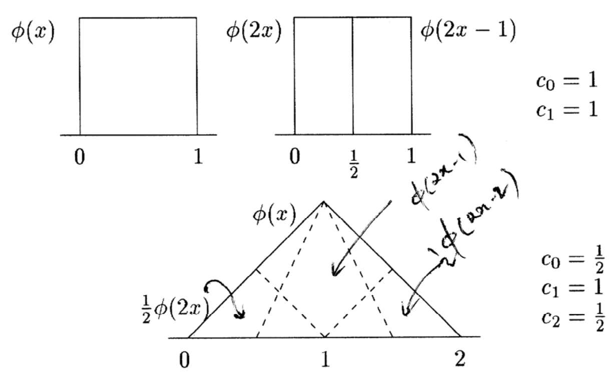
then you will **NOT** get coefficients which produce a reversible transform. Since this is desirable in physics we need to do what Numerical Recipes suggests and force $\sum_k c_k^2 = 1$. This means reducing the coefficients by a further factor $1/\sqrt{2}$. The reason lies buried deep in matrix inversion.

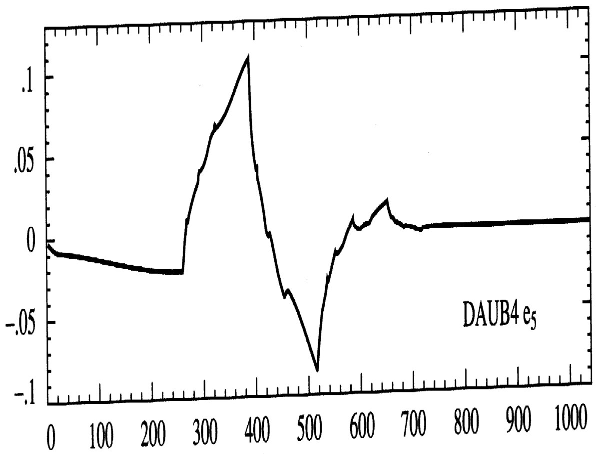
- Here are two examples, our friend Haar and the "top hat"



- Again a $1/\sqrt{2}$ multiplication factor ensures reversibility of the transform
- Ingrid Daubechies invented a four coefficient fractal which is not a simple mother function shape as above, but instead must be constructed by working backwards from the coefficients. They are:

$$c_0 = \tfrac{1}{4}(1+\sqrt{3}), c_1 = \tfrac{1}{4}(3+\sqrt{3}), c_2 = \tfrac{1}{4}(3-\sqrt{3}), c_3 = \tfrac{1}{4}(1-\sqrt{3})$$

- Again "Numerical Recipes" surreptitiously adds a further $1/\sqrt{2}$ and with good reason!



step2: Wavelet functions

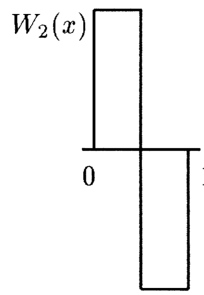
- **From the mother or scaling function and the coefficients we construct wavelet functions $\psi(x)$.**

  $\psi(x) = \sum_k (-1)^k c_{M-k}\phi(2x-k)$ or at other compression levels $\psi(2^{j-1}x) = \sum_k (-1)^k c_{M-k}\phi(2^j x-k)$ with generally $\psi_{jk} = \sum_k (-1)^k c_{M-k}\phi(2^j x-k)$

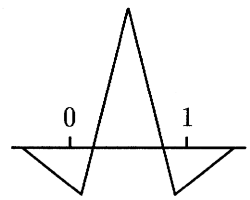
  Three things to note:

  1) The introduction of an alternating negative sign on the coefficients.
  2) The inversion of the order of coefficients assuming there are M coefficients.
  3) If you have multiplied by the requisite fudge factor to get a reversible transform you don't need to do any more on these coefficients.

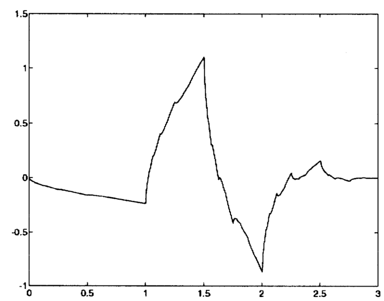
- Here are the basic wavelet shapes at the highest level. The Daubechies wavelet is a construction.



Haar wavelet from box function
$W_2(x) = \phi(2x) - \phi(2x-1)$



"Wavelet" from hat function
$W = \phi(2x) - \frac{1}{2}\phi(2x-1) - \frac{1}{2}\phi(2x+1)$



$W_4(x)$ from $\phi = D_4$ Orthogonal wavelet

Please note the wavelet function for the top hat is strictly $\psi(x) = \frac{1}{2}\phi(2x) - \phi(2x-1) + \frac{1}{2}\phi(2x-2)$ and inverted to the above diagram. The areas sum to zero as does the sum of the coefficients.

step3: Multi Resolution Analysis (MRA)

- Although we have quite general definitions for ϕ_{jk} and ψ_{jk} we need only use the j=0 level over and over again. This was a discovery by Mallet.

Here is the technique:

1) Multiply each a **pair** of input coefficients with the mother function coefficients on the top line and the wavelet coefficients in the bottom line.

Eg: For the non-reversible Haar transform this is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \circ \begin{pmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 4 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 5 \\ 3 \\ 6 \\ 2 \\ 4 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} m \\ w \\ m \\ w \\ m \\ w \\ m \\ w \end{pmatrix}$$

2) Now sort (an effective permutation) the above column matrix and bring all the mother generated coefficients to the top.

$$\begin{pmatrix} 5 \\ 1 \\ 5 \\ 3 \\ 6 \\ 4 \\ 4 \\ 2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 5 \\ 5 \\ 6 \\ 4 \\ 1 \\ 3 \\ 2 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} m \\ m \\ m \\ m \\ w \\ w \\ w \\ w \end{pmatrix}$$

3) Now repeat step 2 only on the coefficients labelled 'm'

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \circ \begin{pmatrix} 5 \\ 5 \\ 6 \\ 4 \\ 1 \\ 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 10 \\ 2 \\ 1 \\ 3 \\ 2 \\ 2 \end{pmatrix}$$

4) repeat step 2) and 3) until only the top coefficient has the 'm' label.

- The complete sequence looks like:

$$\begin{pmatrix}3\\2\\4\\1\\4\\2\\3\\1\end{pmatrix} \overset{T}{\Rightarrow} \begin{pmatrix}5\\1\\5\\3\\6\\2\\4\\2\end{pmatrix} \overset{P}{\Rightarrow} \begin{pmatrix}5\\5\\6\\4\\1\\3\\2\\2\end{pmatrix} \overset{T}{\Rightarrow} \begin{pmatrix}10\\0\\10\\2\\1\\3\\2\\2\end{pmatrix} \overset{P}{\Rightarrow} \begin{pmatrix}10\\10\\0\\2\\1\\3\\2\\2\end{pmatrix} \overset{T}{\Rightarrow} \begin{pmatrix}20\\0\\0\\2\\1\\3\\2\\2\end{pmatrix}$$

- Reversing the above procedure is used to compose the original vector. In this case the multiplying matrix has to be.

$$\begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}$$

which is the inverse of the original coefficient matrix. But this is messy and could be fixed with a universal $1/\sqrt{2}$ to retain the symmetry of the mathematics and is why “Numerical Recipes” adds the factor to the coefficients.

- Check for yourself reverse sequence is:

$$\begin{pmatrix}20\\0\\0\\2\\1\\3\\2\\2\end{pmatrix} \overset{T}{\Rightarrow} \begin{pmatrix}10\\10\\0\\2\\1\\3\\2\\2\end{pmatrix} \overset{P}{\Rightarrow} \begin{pmatrix}10\\0\\10\\2\\1\\3\\2\\2\end{pmatrix} \overset{T}{\Rightarrow} \begin{pmatrix}5\\5\\6\\4\\1\\3\\2\\2\end{pmatrix} \overset{P}{\Rightarrow} \begin{pmatrix}5\\1\\5\\3\\6\\2\\4\\2\end{pmatrix} \overset{T}{\Rightarrow} \begin{pmatrix}3\\2\\4\\1\\4\\2\\3\\1\end{pmatrix}$$

- The Haar Transform is square. If we have multiplying coefficients which do not form a neat square (eg., the Daubechies coefficients) we still use the same technique of producing the ‘m’ and ‘w’ terms. “Numerical Recipes” illustrates how this is done. Also shown is how to handle the end points in this case.

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 & & & & & & & \\ c_3 & -c_2 & c_1 & -c_0 & & & & & & & \\ & & c_0 & c_1 & c_2 & c_3 & & & & & \\ & & c_3 & -c_2 & c_1 & -c_0 & & & & & \\ \vdots & \vdots & & & & & \ddots & & & & \\ & & & & & & & c_0 & c_1 & c_2 & c_3 \\ & & & & & & & c_3 & -c_2 & c_1 & -c_0 \\ c_2 & c_3 & & & & & & & & c_0 & c_1 \\ c_1 & -c_0 & & & & & & & & c_3 & -c_2 \end{bmatrix}$$

- The reverse procedure is created by multiplying pairs by the transpose of the original matrix of coefficients as shown. This also effectively changes the order of the coefficients.

$$\begin{bmatrix} c_0 & c_3 & & & \cdots & & & & c_2 & c_1 \\ c_1 & -c_2 & & & \cdots & & & & c_3 & -c_0 \\ c_2 & c_1 & c_0 & c_3 & & & & & & \\ c_3 & -c_0 & c_1 & -c_2 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & c_2 & c_1 & c_0 & c_3 & \\ & & & & & c_3 & -c_0 & c_1 & -c_2 & \\ & & & & & & & c_2 & c_1 & c_0 & c_3 \\ & & & & & & & c_3 & -c_0 & c_1 & -c_2 \end{bmatrix}$$

- One of the intrinsic advantages of the wavelet transform is that only requires an order(N) computational effort and is much faster than the FFT at vector transformation..

Walsh Function:

The Rademacher functions are an orthogonal system on (0, 1) obtained by adding up all the Haar functions at the same scale. The Rademacher functions were obtained by combining the Haar functions by simply adding them at a given scale. The Walsh functions take sums and differences of the Haar functions to obtain a complete system. We define

$$W_o(t) := \phi(t),\ w_1(t) := \Psi(t),$$

$$W_2(t) := \Psi(2t) + \Psi(2t-1),$$

$W_3(t) := \Psi(2t) - \Psi(2t-1)$,

$W_{2n}(t) := W_n(2t) + W_n(2t-1)$,

$W_{2n+1}(t) := W_n(2t) - W_n(2t-1)$.

(1.6)

Thus these Walsh functions also belong to the wavelet subspaces of the Haar system:

$W_0 \in V_0$, $W_1 \in W_0$, W_2, $W_3 \in W_1$, W_4, W_5, W_6, $W_7 \in W_2$,..................

W_{2m}, W_{2m+1},............................., $W_{2^{m+1}-1} \in W_m$,.......................................

(1.7)

Notice that these defining relations (1.7) are exactly the same as those in the two dilation equations of the Haar system,

$$\phi(t) = \phi(2t) + \phi(2t-1), \tag{1. 8a}$$

$$\Psi(t) = \phi(2t) - \phi(2t-1). \tag{1. 8b}$$

Since all functions defined by (1.8a) are orthogonal to all defined by (1.8b), it follows that W_{2n} and W_{2n+1} are orthogonal. Also if W_n and W_m are orthogonal so are W_{2m}, W_{2m+1}, …, W_{2m+1-1} are orthogonal in W_n. Since all of these functions have support contained in [0,1], the $\{W_n\}$ are an orthogonal system in $L^2(0,1)$. Moreover, there are exactly 2^m Haar functions in W_m whose support lies in [0,1], and therefore the Walsh functions in W_m form a basis of this space. Since the Haar functions are complete in $L^2(0,1)$ so are the Walsh functions. [5]

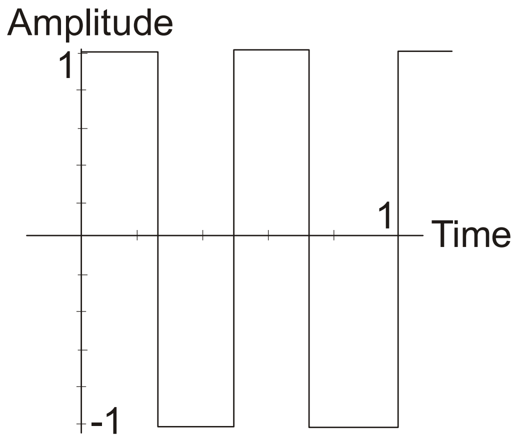


FIGURE 2 One of the Rademacher functions

Comparing Haar Transform with Walsh Transform:

The Haar wavelet packet transform is usually referred to as the *Walsh transform*. A Walsh transform is calculated by performing a 1-level Haar transform on all subsignals, both trends and fluctuations.

For example, consider the signal f defined by

$$f = (4, 6, 8, 10, 12, 14, 16, 18). \tag{1.9}$$

A 1-level Haar transform:

$$\left(\frac{4+6}{\sqrt{2}}, \frac{8+10}{\sqrt{2}}, \frac{12+14}{\sqrt{2}}, \frac{16+18}{\sqrt{2}} \,\middle|\, \frac{4-6}{\sqrt{2}}, \frac{8-10}{\sqrt{2}}, \frac{12-14}{\sqrt{2}}, \frac{16-18}{\sqrt{2}}\right). \tag{1.10}$$

A 1-level Haar transform and a 1-level Walsh transform of f are identical, producing the following signal:

$$\left(5\sqrt{2}, 9\sqrt{2}, 13\sqrt{2}, 17\sqrt{2} \,\middle|\, -\sqrt{2}, -\sqrt{2}, -\sqrt{2}, -\sqrt{2}\right). \tag{1.11}$$

A 2-level Walsh transform is calculated by performing 1-level Haar transforms on both the trend and the fluctuation sub signals, as follows:

$$\left(5\sqrt{2}, 9\sqrt{2}, 13\sqrt{2}, 17\sqrt{2}\right) \rightarrow \left(\frac{5\sqrt{2}+9\sqrt{2}}{\sqrt{2}}, \frac{13\sqrt{2}+17\sqrt{2}}{\sqrt{2}} \,\middle|\, \frac{5\sqrt{2}-9\sqrt{2}}{\sqrt{2}}, \frac{5\sqrt{2}-9\sqrt{2}}{\sqrt{2}}\right),$$

$$\left(-\sqrt{2},-\sqrt{2},-\sqrt{2},-\sqrt{2}\right)\rightarrow\left(\frac{-\sqrt{2}-\sqrt{2}}{\sqrt{2}},\frac{-\sqrt{2}-\sqrt{2}}{\sqrt{2}},\frac{-\sqrt{2}-\left(-\sqrt{2}\right)}{\sqrt{2}},\frac{-\sqrt{2}-\left(-\sqrt{2}\right)}{\sqrt{2}}\right) \quad (1.12)$$

Hence the 2-level Walsh transform of the signal f is the following signal:

$$(14, 30 \mid -4, -4 \mid -2, -2 \mid 0, 0). \quad (1.13)$$

It is interesting to compare this 2-level Walsh transform with the 2-level Haar transform of the signal f. The 2-level Haar transform of f is the following signal :

$$\left(14,30 \middle| \; -4,-4 \middle| \; -\sqrt{2},-\sqrt{2},-\sqrt{2},-\sqrt{2}\right) \quad (1.14)$$

comparing this Haar transform with the Walsh transform in (1.13), we see that the Walsh transform is slightly more compressed in terms of energy, since the last two values of the Walsh transform are zeros. We could, for example, achieve 25 % compression of signal f by discarding the two zeros from its 2-level Walsh transform, but we could not discard any zeros from its 2-level Haar transform. Another advantage of the 2-level Walsh transform is that it is more likely that all of its non-zero values would stand out form a random noise background, because these values have larger magnitudes than the values of the 2-level Haar transform.

A 3-level Walsh transform is performed by calculating 1-level Haar transforms on each of the four sub signals that make up the 2-level Walsh transform. For example, applying 1-level Haar transforms to each of the four sub signals of the 2-level Walsh transform, we obtain

$$\begin{aligned}
&(14,30)\rightarrow(22\sqrt{2} \;\; | -8\sqrt{2}),\\
&(-4,-4)\rightarrow(-4\sqrt{2}|\; 0),\\
&(-2,-2)\rightarrow(-2\sqrt{2}|\; 0),\\
&(0,0)\rightarrow(0|\; 0),
\end{aligned} \quad (1.15)$$

Hence the 3-level Walsh transform of the signal f in (1.5) is :

$$(22\sqrt{2}| \;\; -8\sqrt{2}| \;\; -4\sqrt{2}| \; 0| \;\; -2\sqrt{2}| \; 0| \; 0| \; 0).$$

(1.16)

Here, at the third level, the contrast between the Haar and Walsh transforms is even shaper than at the second level. The 3-level Haar transform of this signal is

$$(22\sqrt{2}| \; -8\sqrt{2}| \; -4,-4| \; -\sqrt{2},-\sqrt{2},-\sqrt{2},-\sqrt{2}), \qquad (1.17)$$

comparing the transforms (1.16) and (1.17) we can see, at least for this particular signal f, that the 3-level Walsh transform achieves a more compact redistribution of the energy of the signal than the Haar transform.

APPLICATIONS OF HAAR TRANSFORMS

In this section we shall discuss two examples of applying wavelet packet transforms to audio and image compression. While wavelet packet transforms can be used for other purposes, such as noise removal, because of space limitations we shall limit our discussion to the arena of compression.

First example, we shall use a Coif 30 wavelet packet transform to compress the audio signal greasy. If we found that a 4-level Coif 30 wavelet transform – with trend values quantized at 8 bpp and fluctuations quantized at 6 bpp, and with separate entropies computed for all sub signals achieved a compression of greasy requiring an estimated 11,305 bits. That is, this compression required an estimated 0.69 bpp (instead of 8 bpp in the original). However, if we use a 4-level Coif 18 wavelet packet transform and quantize in the same way, then the estimated number of bits is 10.158 i.e, 0,62 bpp. This represents a slight improvement over the wavelet transform.

In several respects – in bpp, in RMS Error, and in total number of – significant values – the wavelet packet compression of greasy is nearly as good as or slightly better than the wavelet transform compression. See **Table1.1**

Transform	Sign.Values	Bpp	RMS Error
wavelet	3685	0.69	0.839
w.packet	3072	0.62	0.868

TABLE 1.1 Wavelet and wavelet packet compressions of greasy

Second example, we consider a compression of a fingerprint image. Using the quantizations 9bpp for the trend and 6bpp fort the fluctuations, we obtain an estimated 0.49bpp. That represents a 36 % improvement over the 0,77bpp estimated for the wavelet compression. In **Table 1.2** I show a comparison of these two compressions of Fingerprint 1. Although the wavelet packet transform compression does not produce as small a relative 2-norm error as the wavelet transform compression, nevertheless, a value of 0.043 is still better than the 0.05 rule of thumb value for an acceptable approximation. Taking into account the other data from **Table 1.2** – the number of significant transform values and the number of bpps – it is clear that the wavelet packet compression of Fingerprint 1 is superior to the wavelet compression. [2]

Transform	Sign.Values	Bpp	Rel.2 – norm error
wavelet	33330	0.77	0.35
w.packet	20796	0.49	0.043

TABLE 2: Two compressions of Fingerprint 1

CONTINUOUS WAVELET TRANSFORM

In the continuous wavelet transform, a function Ψ (“psi“), which in practice looks like a little wave, is used to create a family of wavelets Ψ (at + b) where a and b are real number, “a“ dilating (compressing or stretching) the function Ψ and “b“

translating (displacing) it. The word continuous refers to transform, not the wavelets, although people sometimes speak of "continuous wavelets".

The continuous wavelet transform turns a signal f (t) into a function with two variables (scale and time), which one can call c (a,b) :

$$c(a,b) = \int f(t)\Psi(at+b)dt. \qquad (2.1)$$

This transformation is in theory infinitely redundant, but it can be useful in recognizing certain characteristics of o signal. In addition, the extreme redundancy is less of a problem than one might imagine, a number of researchers have found ways of rapidly extracting the essential information from these redundant transforms.

One such method reduces a redundant transform to its skeleton. When certain signals are represented by a continuous wavelet transform, all the significant information of the signal is contained in curves, or "ridges" says Bruno Torréssani of the French Centre National de Recherché Scientifique, who works at the University of Aix – Marseille II. These are essentially the points in the time – frequency plane where the natural frequency of the translated and dilated wavelet coincides with the local frequencies, or one of the local frequencies, of the transform. [3]

Haar wavelet is the simplest wavelet. The Haar wavelet transform, proposed in 1909 by Alfred Haar, is the first known wavelet. Haar transform or Haar wavelet transform has been used as an earliest example for orthonormal wavelet transform with compact support. The Haar wavelet family for $x \in [0, 1]$ is defined as follows:

$$h_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\xi_1, \xi_2), \\ -1 & \text{for} \quad x \in [\xi_2, \xi_3], \\ 0 & \text{elsewhere} \end{cases}$$

where $\xi_1 = \frac{k}{m}, \xi_2 = \frac{k+0.5}{m}$ and $\xi_3 = \frac{k+1}{m}$. In these formulae integer $m = 2^j$, $j = 0, 1, \dots J$ indicates the level of the wavelet; $k = 0, 1, \dots m - 1$ is the translation parameter. Maximal level of resolution is J and 2^J is denoted as $M = 2^J$. The index i is calculated from the formula $i = m + k + 1$; in the case of minimal values $m = 1$, $k = 0$ we have $i = 2$. The maximal value of i is $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1(x) = 1$.

It must be noticed that all the Haar wavelets are orthogonal to each other:

$$\int_0^1 h_i(x)h_l(x)dx = \begin{cases} 2^{-j} & i = l = 2^j + k \\ 0 & i \neq l \end{cases}$$

Therefore, they construct a very good transform basis. Any function $y(x)$, which is square integrable in the interval [0, 1), namely $\int_0^1 y^2(x)dx$ is finite, can be expanded in a Haar series with an infinite number of terms

$$y(x) = \sum_{i=1}^{\infty} c_i h_i(x), \quad i = 2^j + k, \quad j \geqslant 0, \quad 0 \leqslant k \leqslant 2^j, \quad x \in [0, 1)$$

Where the Haar coefficients,

$$c_i = 2^j \int_0^1 y(x)h_i(x)dx$$

are determined in such a way that the integral square error

equation(10)

$$E = \int_0^1 \left[y(x) - \sum_{i=1}^{2M} c_i h_i(x) \right]^2 dx$$

is minimized.

In general, the series expansion of $y(x)$ contains infinite terms. If $y(x)$ is a piecewise constant or may be approximated as a piecewise constant during each subinterval, then $y(x)$ will be terminated at finite terms, that is

$$y(x) \cong \sum_{i=1}^{2M} c_i h_i(x) = c^T h_{2M}(x)$$

where the coefficient and the Haar function vectors are defined as:

$$c^T = [c_1, c_2, \dots c_{2M}], \quad h_{2M}(x) = [h_1(x), h_2(x), \dots h_{2M}(x)]^T$$

respectively and x∈[0,1)x∈[0,1).

The integrals of Haar function $h_i(x)$ can be evaluated as:

$$p_{i,1}(x) = \int_0^x h_i(x)dx$$ pi,1(x)=∫0xhi(x)dx

$$p_{i,v}(x) = \int_0^x p_{i,v-1}(x)dx, \quad v = 2, 3, \dots$$

Carrying out these integrations with the aid, it is found that

$$p_{i,1}(x) = \begin{cases} x - \xi_1 & \text{for} \quad x \in [\xi_1, \xi_2], \\ \xi_3 - x & \text{for} \quad x \in [\xi_2, \xi_3], \\ 0 & \text{elsewhere} \end{cases}$$

$$p_{i,2}(x) = \begin{cases} 0 & \text{for} \quad x \in [0, \xi_1], \\ \frac{(x-\xi_1)^2}{2} & \text{for} \quad x \in [\xi_1, \xi_2], \\ \frac{1}{4m^2} - \frac{(\xi_3 - x)^2}{2} & \text{for} \quad x \in [\xi_2, \xi_3], \\ \frac{1}{4m^2} & \text{for} \quad x \in [\xi_3, 1] \end{cases}$$

$$p_{i,3}(x) = \begin{cases} 0 & \text{for} \quad x \in [0, \xi_1], \\ \frac{(x-\xi_1)^3}{6} & \text{for} \quad x \in [\xi_1, \xi_2], \\ \frac{x-\xi_2}{4m^2} - \frac{(\xi_3 - x)^3}{6} & \text{for} \quad x \in [\xi_2, \xi_3], \\ \frac{x-\xi_2}{4m^2} & \text{for} \quad x \in [\xi_3, 1] \end{cases}$$

$$p_{i,4}(x) = \begin{cases} 0 & \text{for} \quad x \in [0, \xi_1], \\ \frac{(x-\xi_1)^4}{24} & \text{for} \quad x \in [\xi_1, \xi_2], \\ \frac{(x-\xi_2)^2}{8m^2} - \frac{(\xi_3-x)^4}{24} + \frac{1}{192m^4} & \text{for} \quad x \in [\xi_2, \xi_3], \\ \frac{(x-\xi_2)^2}{8m^2} + \frac{1}{192m^4} & \text{for} \quad x \in [\xi_3, 1] \end{cases}$$

Let us define the collocation points $x_l = (l - 0.5)/(2M)$, $l = 1, 2, \ldots, 2M$. By these collocation points, a discretizised form of the Haar function $h_i(x)$ can be obtained. Hence, the coefficient matrix $H(i, l) = (h_i(x_l))$, which has the dimension $2M \times 2M$, is achieved. The operational matrices of integrations $P\upsilon$, which are $2M$ square matrices, are defined by the equation $P\upsilon(i, l) = p_{i,\upsilon}(x_l)$, where υ shows the order of integration.

CONCLUSION:

In this paper, Haar equation is proposed for the generalized wavelet transformation equation. Comparisons of the haar and wavelet transformation show that our method is efficient method. These calculations demonstrate that the accuracy of the Haar wavelet solutions is quite high even in the case of a small number of grid points. Applications of this method are very simple, and also it gives the implicit form of the approximate solutions of the problems. These are the main advantages of the method. Hence, the present method is a very reliable, simple, fast, minimal computation costs, flexible, and convenient alternative method.

REFERENCES

- Walter, Gilbert G. & Shen, Xiaoping, Wavelets and Other Orthogonal Systems, Chapman & Hall / CRC, 2001.

- Walker, J.S., A Primer on Wavelets and Their Scientific Applications, Printice-Hall Inc., 1999.
- Resnikoff, Howard L. & Wells, Raymond O., Wavelet Analysis, Springer-Verlag New York Inc., 1998.
- Hubband, Barbara Burke, The World According to Wavelets, AK Peters Ltd., 1995.
- Vetterli, Martin & Kovacevic, Jelena, Wavelets and Subband Coding, Printice Hall Inc., 1995.
- Benedetto, John J. & Frazier, Micheal W., Wavelets : Mathematics and Applications, CRC Press Inc., 1994.
- 7. Daubechies, Ingrid, Ten Lectures on Wavelets, Capital City Press, 1992.
- Bredon, Glen E. "A new treatment of the Haar integral." *The Michigan Mathematical Journal* 10.4 (1963): 365-373.
- Dubins, Lester E. "Measurable tail disintegrations of the Haar integral are purely finitely additive." *Proceedings of the American Mathematical Society* 62, no. 1 (1977): 34-36.
- Williams, David, and John F. Cornwell. "The Haar integral for Lie supergroups." *Journal of mathematical physics* 25, no. 10 (1984): 2922-2932.