International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 14
October2016

A Study On Problem Solving Using Bayes Theorem

Ismael Yaseen Abdulridha Alasadi

M.Sc, Applied Mathematics

University College of Science, Osmania University, India

University of Dhi qar, Iraq

Email: 7207183617ismael@gmail.com

ABSTRACT:

The study on understanding of Bayes' Theorem and to use that knowledge to investigate practical problems in various professional fields. Provides a means for making probability calculations after revising probabilities when obtaining new information in an important phase of probability analysis. When given P(A) and P(A∩B), one can calculate P(B/A) by manipulating the information in the Multiplication Rule. However, one could not calculate P(A/B). Similarly, when given P(B) and P(A∩B), one can calculate P(A/B) by manipulating the information in the Multiplication Rule. There is where one can now apply Bayes' Theorem.

Keywords: Bayes' Theorem, Multiplication Rule, Probabilities.

INTRODUCTION:

Bayes' theorem is a basic element of probability theory first discovered or codified by the British statistician, Thomas Bayes. In probability theory and statistics,

International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 14
October2016

Bayes’ theorem describes the probability of an event, based on conditions that might be related to the event. For example, if cancer is related to age, then, using Bayes’ theorem, a person’s age can be used to more accurately assess the probability that they have cancer. One of the many applications of Bayes’ theorem is Bayesian inference, a particular approach to statistical inference. When applied, the probabilities involved in Bayes’ theorem may have different probability interpretations. With the Bayesian probability interpretation the theorem expresses how a subjective degree of belief should rationally change to account for evidence. Bayesian inference is fundamental to Bayesian statistics.

Bayes’ theorem is named after Rev. Thomas Bayes who first provided an equation that allows new evidence to update beliefs. It was further developed by Pierre-Simon Laplace, who first published the modern formulation in his 1812 “Théorie analytique des probabilités.” Sir Harold Jeffreys put Bayes’ algorithm and Laplace's formulation on an axiomatic basis. Jeffreys wrote that Bayes’ theorem “is to the theory of probability what the Pythagorean theorem is to geometry.”

At the most basic level, Bayes' theorem is an equation that relates two conditional probabilities, P(B|A) and P(A|B).

$$P(B|A) = \frac{P(B)P(A|B)}{P(A)}$$

This formula is easily derived from the definition of conditional probability and the multiplication rule. (In fact, it would be a fairly straightforward homework assignment.) An intermediate step in the derivation is the equation:

$$P(B|A) = \frac{P(A \cap B)}{P(A \cap B) + P(A \cap {\sim}B)}$$

Despite its simplicity, from Bayes' Theorem a whole new approach to statistics develops.

Importance of Bayes' Theorem

It could rightly be said that today the field of statistics is being gradually transformed by

Bayes' theorem, producing a new field of **Bayesian statistics**. This revolution not only has immense potential for scientific research, but for fundamentally changing how probabilistic thinking occurs in human culture. Bayesian statistics:

- Is arguably a superior way of thinking about and using probabilities.
- Has the potential to transform every discipline that draws inferences from uncertain evidence (medicine, law, quality analysis…)
- Helps us to see that many important inferences are only probabilistic, not certain
- Requires us to identify and explicitly quantify the components of probabilistic inference.
- Makes everyday probabilistic inferences less vague and subjective

Bayes' Theorem and Scientific Inference

While we cannot go into much detail here, a simple example helps illustrate why Bayesian statistics is of such fundamental importance.

Let:

H = a scientific hypothesis (e.g., about a population mean)

E = observed evidence (e.g. a sample mean).

It should be obvious that the purpose of scientific research is to collect evidence in order to evaluate a hypothesis. In other words, our principle concern is to estimate the probability that a hypothesis is true, given empirical evidence, or $P(H \mid E)$.

Yet conventional statistical theory examines the reverse question: what is the probability of the evidence we've observed occurring, given some hypothesis, or P(E|H). For example, the whole system of statistical significance testing is built around estimating the likelihood of observed results given a null hypothesis. This is completely

International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 14
October2016

backwards. It means we're estimating the probability of something we already know (our data), given something we don't know (our hypothesis). It is this logical backwardness that makes null hypothesis testing and p-values so non-intuitive and confusing.

Clearly the correct order of scientific inference is to estimate what we don't already know (our hypothesis), given what we do know (our evidence).

But by Bayes' theorem, we see that these two conditional probabilities are related:

$$P(H|E) = \frac{P(H)P(E|H)}{P(E)}$$

And this, more or less, is what opens up the whole realm of Bayesian statistics.

Many advanced applications (e.g., Google spam filters and language translators) already use Bayesian methods. Bayesian methods will likely become more and more common in the coming decades.

BAYES' THEOREM PROBLEMS AND SOLUTIONS:

Example 1:

B: test (positive) A: no AIDS

B^c : test (negative) A^c : AIDS

From past experience and records, we know

$$P(A) = 0.99, P(B \mid A) = 0.03, P(B \mid A^c) = 0.9$$

That is, we know the probability of a patient having no AIDS, the conditional probability of test positive given having no AIDS (wrong diagnosis), and the conditional probability of test positive given having AIDS (correct diagnosis).

Our object is to find $P(A \mid B)$, i.e., we want to know the probability of a patient having not AIDS even known that this patient is test positive.

Example 2:

A_1 : the finance of the company being good.

A_2 : the finance of the company being O.K.

International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 14
October2016

A_3 : the finance of the company being bad.

B_1 : good finance assessment for the company.

B_2 : O.K. finance assessment for the company.

B_3 : bad finance assessment for the company.

From the past records, we know

$$P(A_1)=0.5, P(A_2)=0.2, P(A_3)=0.3,$$
$$P(B_1 \mid A_1)=0.9, P(B_1 \mid A_2)=0.05, P(B_1 \mid A_3)=0.05.$$

That is, we know the chances of the different finance situations of the company and

the conditional probabilities of the different assessments for the company given the finance of the company known, for example, $P(B_1 \mid A_1)=0.9$ indicates 90% chance of good finance year of the company has been predicted correctly by the finance assessment.

Our objective is to obtain the probability $P(A_1 \mid B_1)$, i.e., the conditional probability that the finance of the company being good in the coming year given that good finance assessment for the company in this year.

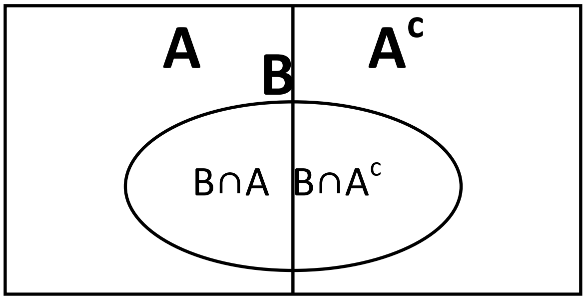
To find the required probability in the above two examples, the following Bayes's theorem can be used.

Bayes's Theorem (two events):

$$P(A \mid B)=\frac{P(A \cap B)}{P(B)}=\frac{P(A)P(B \mid A)}{P(A)P(B \mid A)+P(A^c)P(B \mid A^c)}$$

[Derivation of Bayes's theorem (two events)]:

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 14
October2016



We want to know $P(A \mid B) = \dfrac{P(B \cap A)}{P(B)}$. Since

$$P(B \cap A) = P(A)P(B \mid A),$$

and

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(A)P(B \mid A) + P(A^c)P(B \mid A^c),$$

thus,

$$P(A \mid B) = \frac{P(B \cap A)}{P(B)} = \frac{P(B \cap A)}{P(B \cap A) + P(B \cap A^c)} = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A^c)P(B \mid A^c)}$$

Example 3:

$P(A^c) = 1 - P(A) = 1 - 0.99 = 0.01$. Then, by Bayes's theorem,

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A^c)P(B \mid A^c)} = \frac{0.99 * 0.03}{0.99 * 0.03 + 0.01 * 0.98} = 0.7519$$

$\Rightarrow$ **A patient with test positive still has high probability (0.7519) of no AIDS.**

Bayes's Theorem (general):

Let $A_1, A_2, \ldots, A_n$ be mutually exclusive events and

$$A_1 \cup A_2 \cup \cdots A_n = S,$$

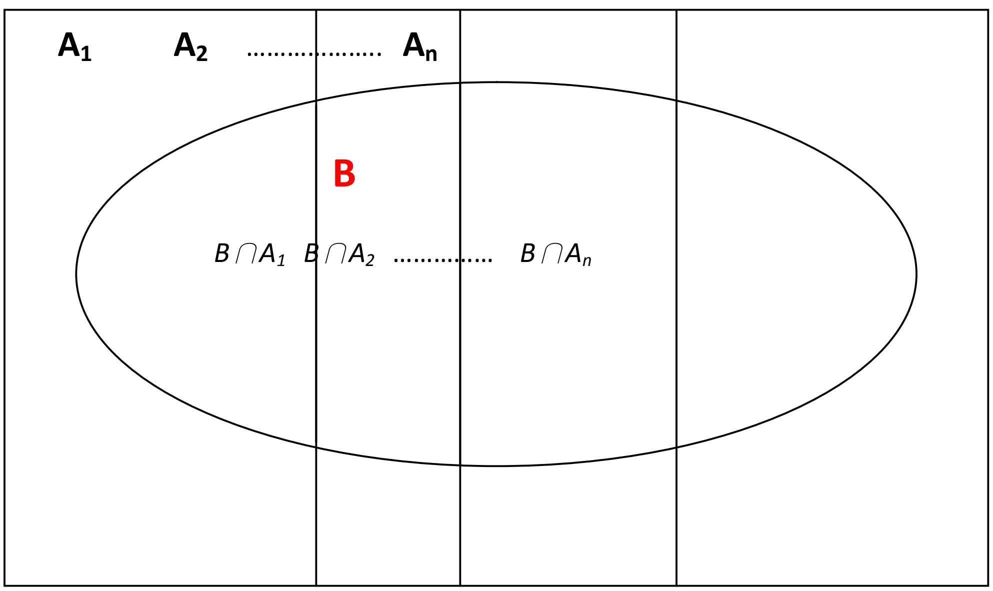
then

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

$$\ldots\ldots\ldots\ldots = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)},$$

$i = 1, 2, \ldots, n$.

[Derivation of Bayes's theorem (general)]:



Since

$$P(B \cap A_i) = P(A_i)P(B \mid A_i),$$

and

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots P(B \cap A_n)$$
$$\ldots\ldots = P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)\text{'}$$

thus,

$$P(A_i \mid B) = \frac{P(B \cap A_i)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)}$$

Example 4:

$$P(A_1 \mid B_1) = \frac{P(A_1)P(B_1 \mid A_1)}{P(A_1)P(B_1 \mid A_1) + P(A_2)P(B_1 \mid A_2) + P(A_3)P(B_1 \mid A_3)}$$
$$\ldots\ldots\ldots\ldots\ldots = \frac{0.5 * 0.9}{0.5 * 0.9 + 0.2 * 0.05 + 0.3 * 0.05} = 0.95$$

$\Rightarrow$ **A company with good finance assessment has very high probability (0.95) of good finance situation in the coming year.**

Example 5:

In a recent survey in a Statistics class, it was determined that only 60% of the students attend class on Thursday. From past data it was noted that 98% of those who went to class on Thursday pass the course, while only 20% of those who did not go to class on Thursday passed the course.

(a) What percentage of students is expected to pass the course?

(b) Given that a student passes the course, what is the probability that he/she attended classes on Thursday.

[solution:]

A1: the students attend class on Thursday

A2: the students do ***not*** attend class on Thursday $\Rightarrow A1 \cup A2 = \Omega$

B1: the students pass the course

B2: the students do ***not*** pass the course

$$P(A_1) = 0.6,\ P(A_2) = 1 - P(A_1) = 0.4,\ P(B_1 \mid A_1) = 0.98,\ P(B_1 \mid A_2) = 0.2$$

(a)

$$\begin{aligned} P(B_1) &= P(B_1 \cap A_1) + P(B_1 \cap A_2) \\ &= P(A_1)P(B_1 \mid A_1) + P(A_2)P(B_1 \mid A_2) \\ &= 0.6 \cdot 0.98 + 0.4 \cdot 0.2 \\ &= 0.668 \end{aligned}$$

(b) By Bayes' theorem,

$$\begin{aligned} P(A1 \mid B_1) &= \frac{P(A_1 \cap B_1)}{P(B_1)} = \frac{P(A1)P(B_1 \mid A1)}{P(A1)P(B_1 \mid A1) + P(A2)P(B_1 \mid A2)} \\ &= \frac{0.6 \cdot 0.98}{0.6 \cdot 0.98 + 0.4 \cdot 0.2} \\ &= 0.854 \end{aligned}$$

PRACTICAL PROBLEMS:

A particular test correctly identifies those with a certain serious disease 94% of the time and correctly diagnoses those without the disease 98% of the time. A friend has just informed you that he has received a positive result and asks for your advice about how to interpret these probabilities. He knows nothing about probability, but he feels that because the test is quite accurate, the probability that he does have the disease is quite high, likely in the 95% range. You want to use your knowledge of probability to address your friend's concerns. What is the probability your friend actually has the disease? We'll tackle this problem a little later using Bayes' Theorem. Right now, let's focus our attention on ideas that lead us to Bayes' Theorem. Specifically, we'll look at conditional probability and the multiplication rule for two dependent events.

The **conditional probability** of an event B in relationship to an event A is the probability that event B occurs after event A has already occurred.

We denote "probability of event B given event A has occurred" by: $P(B|A)$

Multiplication Rule (two dependent events):

$$P(A \text{ and } B) = P(A)\cdot P(B|A) = P(B \text{ and } A) = P(B)\cdot P(A|B)$$

The multiplication rule gives us a method for finding the probability that events A and B both occur, as illustrated by the next two examples.

Example 1:

In a class with 3/5 women and 2/5 men, 25% of the women are business majors. Find the probability that a student chosen from the class at random is a female business major.

Define the relevant events: W = the student is a woman

B = the student is a business major

Express the given information and question in probability notation:

"class with 3/5 women" $\Rightarrow P(W) = 3/5 = 0.60$

"25% of the women are business majors" is the same as saying "the probability a student is a business major, given the student is a woman is 0.25" $\Rightarrow P(B|W) = 0.25$

"probability that a student chosen from the class at random is a female business major" is the same as saying "probability student is a woman and a business major" $\Rightarrow P(W \text{ and } B)$

Use the multiplication rule to answer the question:

$$P(W \text{ and } B) = P(W)\cdot P(B|W) = (0.60)(0.25) = 0.15$$

Example 2:

International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 14
October2016

A box contains 5 red balls and 9 green balls. Two balls are drawn in succession **without** replacement. That is, the first ball is selected and its color is noted but it is not replaced, then a second ball is selected. What is the probability that:

a. the first ball is green and the second ball is green?

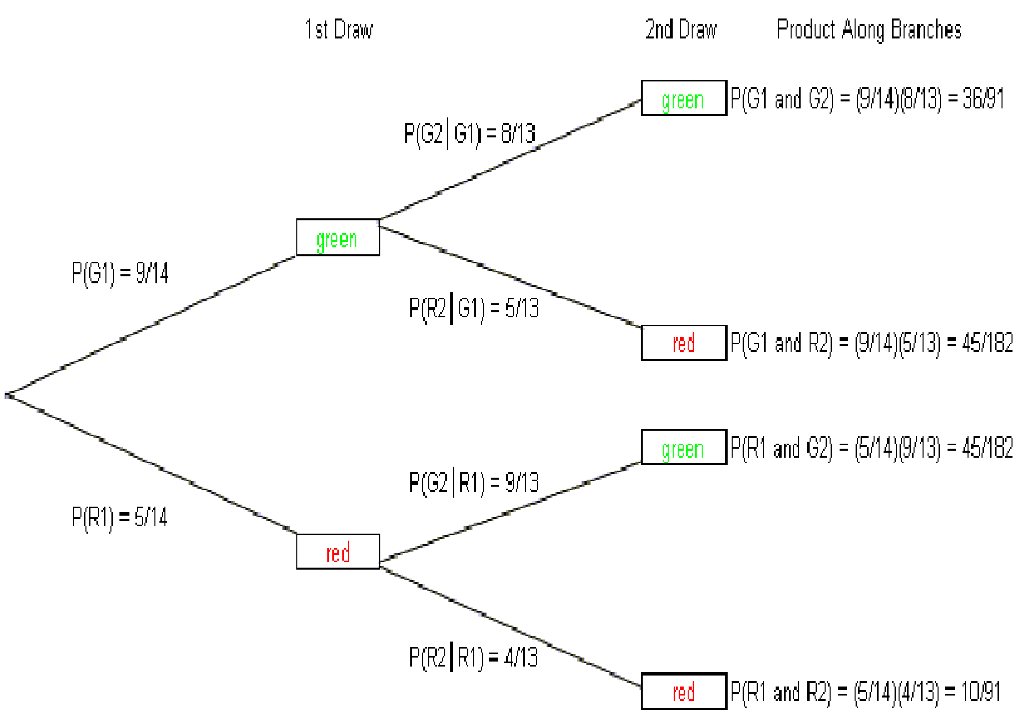
b. the first ball is green and the second ball is red?

c. the first ball is red and the second ball is green?

d. the first ball is red and the second ball is red?

Solutions:

We will construct a tree diagram to help us answer these questions.



Using the tree diagram, we see that:

a. the probability the first ball is green and the second ball is green = $P(G1 \text{ and } G2) = \frac{36}{91}$

b. the probability the first ball is green and the second ball is red = $P(G1 \text{ and } R2) = \frac{45}{182}$

c. the probability the first ball is red and the second ball is green = $P(R1 \text{ and } G2) = \frac{45}{182}$

d. the probability the first ball is red and the second ball is red = $P(R1 \text{ and } R2) = \frac{10}{91}$

Formula for Conditional Probability:

The probability that the second event B occurs given that the first event A has occurred can be found by:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}, \quad \text{where } P(A) \neq 0$$

Note: This formula is obtained from the Multiplication Rule for two dependent events.

(Using algebra, we solve for $P(B \mid A)$ by dividing both sides of the equation by $P(A)$)

The key to solving conditional probability problems is to:

1. Define the events.
2. Express the given information and question in probability notation.
3. Apply the formula.

Example 3:

The probability that Sam parks in a no-parking zone and gets a parking ticket is 0.06. The probability that Sam has to park in a no-parking zone (he cannot find a legal parking space) is

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 14
October2016

0.20. Today, Sam arrives at school and has to park in a no-parking zone. What is the probability that he will get a parking ticket?

Solution:

Define the events: N = Sam parks in a no-parking zone, T = Sam gets a parking ticket

Express the given information and question in probability notation:

"probability that Sam parks in a no-parking zone and gets a parking ticket is 0.06"

tells us that $P(N \text{ and } T) = 0.06$.

"probability Sam has to park in the no-parking zone is 0.20" tells us that $P(N) = 0.20$

"Today, Sam arrives at school and has to park in a no-parking zone. What is the probability that he will get a parking ticket?" is the same as "What is the probability he will get a parking ticket, given that he has to park in a no-parking zone" That is, we want to find $P(T|N)$.

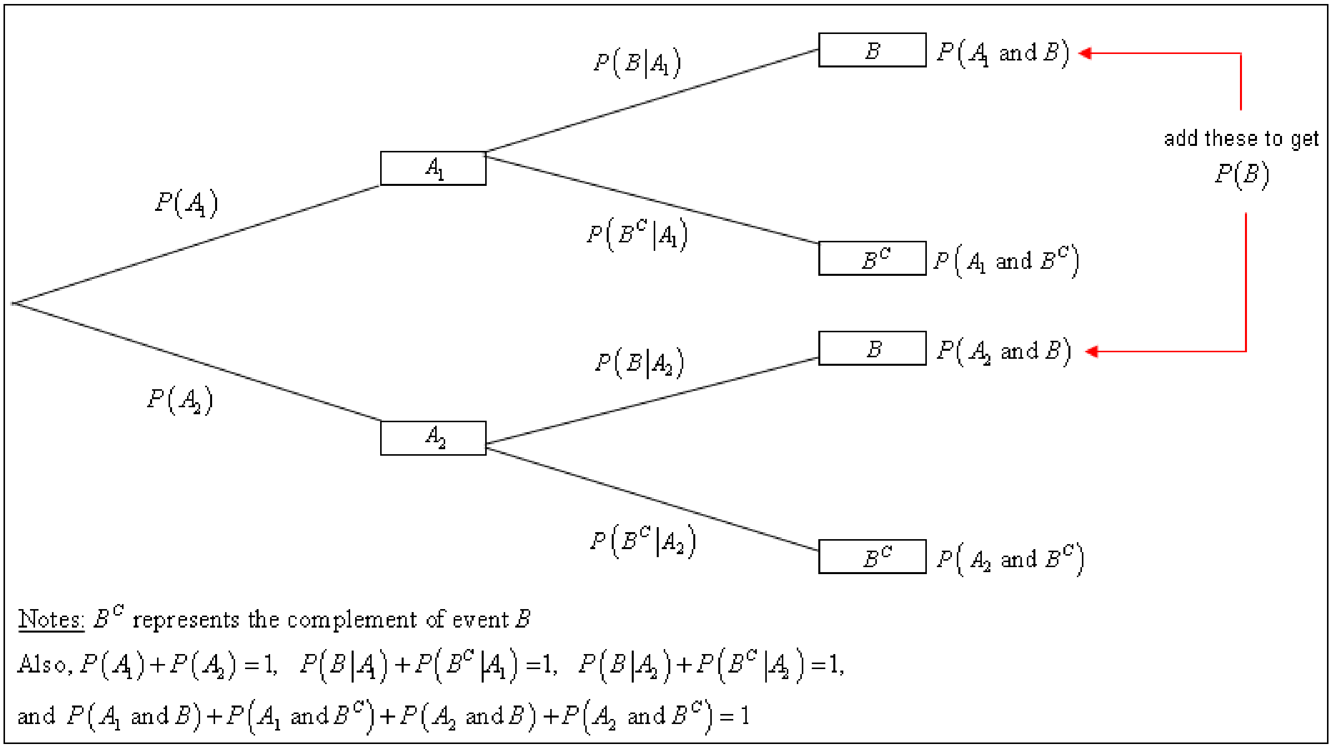
Apply the formula: $P(T|N) = \dfrac{P(N \text{ and } T)}{P(N)} = \dfrac{0.06}{0.20} = 0.30$

Note: Students seem to have difficulties understanding that the question asks us to find $P(T|N)$, not $P(N \text{ and } T)$. They think the answer is 0.06. They fail to consider that Sam could park in a no-parking zone but not receive a ticket. It might be useful to construct a Venn diagram.

The Law of Total Probability:

If A_1 and A_2 are mutually exclusive events with $P(A_1) + P(A_2) = 1$, then for any event B,

$$\begin{aligned} P(B) &= P(A_1 \text{ and } B) + P(A_2 \text{ and } B) \\ &= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) \end{aligned}$$



More generally, if $A_1, A_2, \ldots, A_k$ are mutually exclusive events with $P(A_1)+P(A_2)+\cdots+P(A_k)=1$, then for any event B,

$$P(B) = P(A_1 \text{ and } B) + P(A_2 \text{ and } B) + \cdots + P(A_k \text{ and } B)$$
$$= P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + \cdots + P(A_k)\cdot P(B|A_k)$$

Example 4: An automobile dealer has kept records on the customers who visited his showroom. Forty percent of the people who visited his dealership were women. Furthermore, his records show that 37% of the women who visited his dealership purchased an automobile, while 21% of the men who visited his dealership purchased an automobile.

a. What is the probability that a customer entering the showroom will buy an automobile?

b. Suppose a customer visited the showroom and purchased a car. What is the probability that the customer was a woman?

International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 14
October2016

c. Suppose a customer visited the showroom but did not purchase a car. What is the probability that the customer was a man?

Define the events: $A_1 =$ customer is a woman
$A_2 =$ customer is a man
$B =$ customer purchases an automobile
$B^C =$ customer does not purchase an automobile

Express the given information and question in probability notation:

"Forty percent of the people who visited his dealership were women" $\Rightarrow P(A_1) = 0.40$

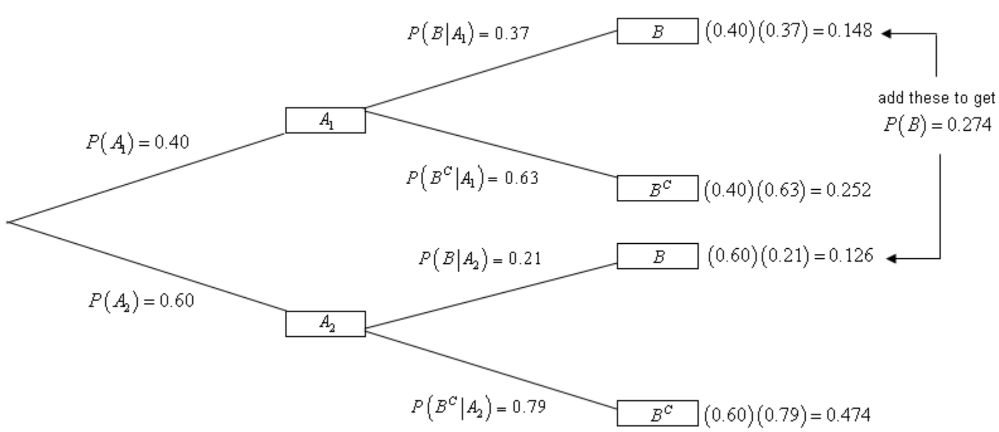
this statement also tells us that 60% of the customers must be men $\Rightarrow P(A_2) = 0.60$

"37% of the women who visited his dealership purchased an automobile" $\Rightarrow P(B|A_1) = 0.37$

"21% of the men who visited his dealership purchased an automobile" $\Rightarrow P(B|A_2) = 0.21$

"What is the probability that a customer entering the showroom will buy an automobile?" $\Rightarrow P(B) = ?$

Create a tree diagram:



Use your tree diagram and the Law of Total Probability to answer the question:

$P(B) = 0.274$

Solution to part b:

"Suppose a customer visited the showroom and purchased a car. What is the probability that the customer was a woman?"

Express the question in probability notation:

We can rewrite the question as, "What is the probability that the customer was a woman, *given* that the customer purchased an automobile." That is, we want to find $P(A_1|B)$

We can use Bayes' Theorem to help us compute this conditional probability.

Bayes' Theorem (Two-Event Case):

$$P(A_1|B)=\frac{P(A_1)\cdot P(B|A_1)}{P(A_1)\cdot P(B|A_1)+P(A_2)\cdot P(B|A_2)}=\frac{P(A_1\text{ and }B)}{P(A_1\text{ and }B)+P(A_2\text{ and }B)}=\frac{P(A_1\text{ and }B)}{P(B)}$$

where A_1 and A_2 are mutually exclusive events with $P(A_1)+P(A_2)=1$
and B is any event with $P(B)\neq 0$

Use Bayes' Theorem and your tree diagram to answer the question:

$$P(A_1|B)=\frac{P(A_1)\cdot P(B|A_1)}{P(A_1)\cdot P(B|A_2)+P(A_2)\cdot P(B|A_2)}=\frac{P(A_1\text{ and }B)}{P(A_1\text{ and }B)+P(A_2\text{ and }B)}=\frac{0.148}{0.148+0.126}=\frac{0.148}{0.274}\approx 0.540$$

The probabilities needed for the computation are easily obtained from our tree diagram.

We already found $P(A_1\text{ and }B)+P(A_2\text{ and }B)$, which is $P(B)$, for part a.) of this example and $P(A_1\text{ and }B)$ is obtained by following the tree diagram path $\rightarrow A_1 \rightarrow B$, the product of the corresponding probabilities is 0.148.

Solution to part c:

"Suppose a customer visited the showroom but did not purchase a car. What is the probability that the customer was a man?

Express the question in probability notation:

We can rewrite the question as, "What is the probability that the customer was a man, *given* that the customer did not purchase an automobile." That is, we want to find $P\left(A_2 \middle| B^C\right)$

Use Bayes' Theorem and your tree diagram to answer the question:

$$P\left(A_2 \middle| B^C\right) = \frac{P(A_2)\cdot P\left(B^C \middle| A_2\right)}{P(A_2)\cdot P\left(B^C \middle| A_2\right) + P(A_1)\cdot P\left(B^C \middle| A_1\right)} = \frac{0.474}{0.474+0.252} = \frac{0.474}{0.726} \approx 0.653$$

Again, the probabilities needed for the computation are easily obtained from our tree diagram.

Additional Notes:

The probabilities $P(A_1)$ and $P(A_2)$ are called prior probabilities because they are initial or *prior* probability estimates for specific events of interest. When we obtain new information about the events we can update the prior probability values by calculating revised probabilities, referred to as posterior probabilities. The conditional probabilities $P\left(A_1 \middle| B\right)$, $P\left(A_2 \middle| B\right)$, $P\left(A_1 \middle| B^C\right)$, and $P\left(A_2 \middle| B^C\right)$ are posterior probabilities. Bayes' Theorem enables us to compute these posterior probabilities.

Example 5:

Let's return to the scenario that began our discussion: A particular test correctly identifies those with a certain serious disease 94% of the time and correctly diagnoses those without the disease 98% of the time. A friend has just informed you that he has received a positive result and asks for your advice about how to interpret these probabilities. He knows nothing about probability, but he feels that because the test is quite accurate, the probability that he does have the disease is quite high, likely in the 95% range. Before attempting to address your friend's concern, you research the illness and discover that 4% of men have this disease. What is the probability your friend actually has the disease?

Define the events: $A_1 =$ a man has this disease
$A_2 =$ a man does not have this disease
$B =$ positive test result
$B^C =$ negative test result

Express the given information and question in probability notation:

"test correctly identifies those with a certain serious disease 94% of the time" $\Rightarrow P\left(B \mid A_1\right) = 0.94$

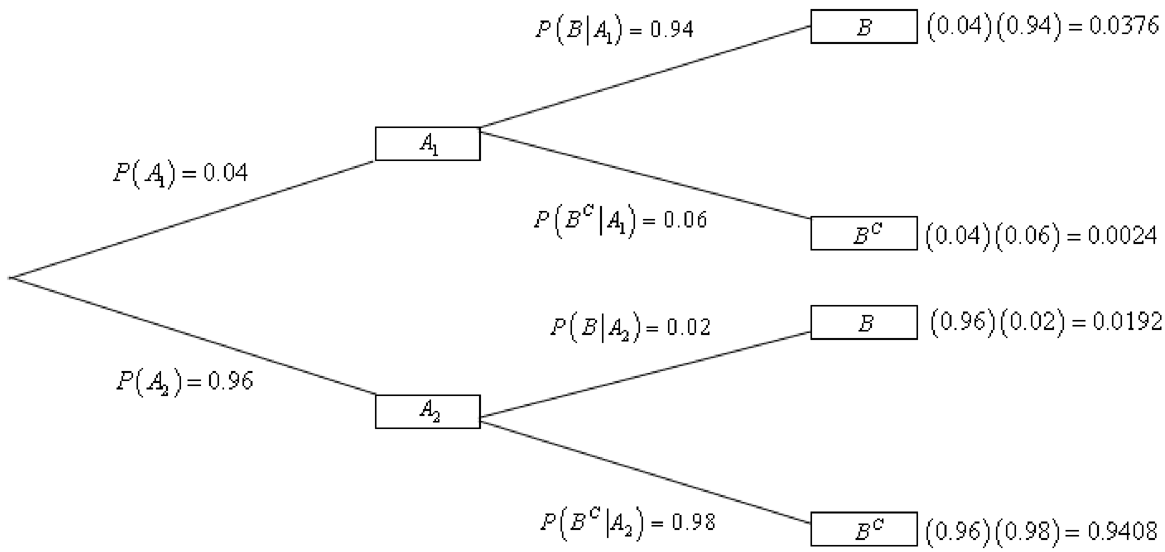
"test correctly diagnoses those without the disease 98% of the time" $\Rightarrow P\left(B^C \middle| A_2\right) = 0.98$

"you discover that 4% of men have this disease" $\Rightarrow P\left(A_1\right) = 0.04$

this statement also tells us that 96% of men do not have the disease $\Rightarrow P\left(A_2\right) = 0.96$

"What is the probability your friend actually has the disease (*given* a positive result)?" $\Rightarrow P\left(A_1 \middle| B\right) = ?$

Construct a tree diagram:



Use Bayes' Theorem and your tree diagram to answer the question:

$$P(A_1|B) = \frac{P(A_1)\cdot P(B|A_2)}{P(A_1)\cdot P(B|A_2) + P(A_2)\cdot P(B|A_2)} = \frac{0.0376}{0.0376 + 0.0192} \approx 0.662$$

There is a 66.2% probability that he actually has the disease. The probability is high, but considerably lower than your friend feared.

CONCLUSION:

Bayes' theorem offers a natural way to unfold experimental distributions in order to get the best estimates of the true ones. The weak point of the Bayes approach, namely the need of the knowledge of the initial distribution, can be overcome by an iterative procedure. Since the method proposed here does not make use of continuous variables, but simply of cells in the spaces of the true and of the measured quantities, it can be applied in multidimensional problems. Although the derivation for Bayes' theorem is straightforward, not everyone is comfortable with it. The difficult aspect to accept is that instead of using probability to predict the future, you are using it to make inferences about the past. People who think in terms of causality have trouble with this.

REFERENCE:

- Elstein, A. S., & Schwarz, A. (2002). Clinical problem solving and diagnostic decision making: selective review of the cognitive literature. *British Medical Journal*, *324*(7339), 729.
- Lindley, D. V. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, 102-107.
- D'Agostini, G. (1995). A multidimensional unfolding method based on Bayes' theorem. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, *362*(2), 487-498.
- Saaty, T. L., & Vargas, L. G. (1998). Diagnosis with dependent symptoms: Bayes theorem and the analytic hierarchy process. *Operations Research*, *46*(4), 491-502.
- Azoury, K. S. (1985). Bayes solution to dynamic inventory models under unknown demand distribution. *Management science*, *31*(9), 1150-1160.
- Fagan, T. J. (1975). Letter: nomogram for Bayes theorem. *The New England journal of medicine*, *293*(5), 257-257.
- Pawlak, Z. (2002). Rough sets, decision algorithms and Bayes' theorem. *European Journal of*

Operational Research, *136*(1), 181-189.

- Moons, K. G., van Es, G. A., Deckers, J. W., Habbema, J. D. F., & Grobbee, D. E. (1997). Limitations of sensitivity, specificity, likelihood ratio, and Bayes' theorem in assessing diagnostic probabilities: a clinical example. *Epidemiology*, 12-17.
- Smets, P. (1993). Belief functions: the disjunctive rule of combination and the generalized Bayesian theorem. *International Journal of approximate reasoning*, *9*(1), 1-35.
- Rudner, L. M., & Liang, T. (2002). Automated essay scoring using Bayes' theorem. *The Journal of Technology, Learning and Assessment*, *1*(2).
- Koch, K. R. (1990). Bayes' Theorem. In *Bayesian Inference with Geodetic Applications* (pp. 4-8). Springer Berlin Heidelberg.