Analysis of Lebesgue Integral Functions and Theorems

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ABSTRACT:

In this article, we define the integral of real-valued functions on an arbitrary measure space and derive some of its basic properties. We refer to this integral as the Lebesgue integral, whether or not the domain of the functions is subset of equipped with Lebesgue measure. The Lebesgue integral applies to a much wider class of functions than the Riemann integral and is better behaved with respect to point wise convergence.

Keywords: Integral, Lebesgue integral, Riemann integral, Real-valued functions

INTRODUCTION:

In mathematics, the integral of a non-negative function of a single variable can be regarded, in the simplest case, as the area between the graph of that function and the x-axis. The Lebesgue integral extends the integral to a larger class of functions. It also extends the domains on which these functions can be defined.

Mathematicians had long understood that for non-negative functions with a smooth enough graph—such as continuous functions on closed bounded intervals—the area under the curve could be defined as the integral, and computed using approximation techniques on the region by polygons. However, as the need to consider more irregular functions arose—e.g., as a result of the limiting processes of mathematical analysis and the mathematical theory of probability—it

became clear that more careful approximation techniques were needed to define a suitable integral. Also, one might wish to integrate on spaces more general than the real line. The Lebesgue integral provides the right abstractions needed to do this important job. The term Lebesgue integration can mean either the general theory of integration of a function with respect to a general measure, as introduced by Lebesgue, or the specific case of integration of a function defined on a sub-domain of the real line with respect to Lebesgue measure.

In this study, we shall define the integral of a function on Rn, in a progressive way, with increasing order of complexity. Before we do so, we shall state some facts about Riemann integrability in measure theoretic language.

INTEGRAL:

Integration is a core concept of advanced mathematics, specifically in the fields of infinitesimal calculus and mathematical analysis. Given a function $f(x)$ of a real variable x and an interval $[a,b]$ of the real line, the integral

$$\int_a^b f(x)\, dx \,,$$

is equal to the area of a region in the xy-plane bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$, with areas below the x-axis being subtracted.

The term "integral" may also refer to the notion of antiderivative, a function F whose derivative is the given function f. In this case it is called an indefinite integral, while the integrals discussed in this article are termed definite integrals. Some authors maintain a distinction between antiderivatives and indefinite integrals.

The principles of integration were formulated by Isaac Newton and Gottfried Leibniz in the late seventeenth century. Through the fundamental theorem of calculus, which they independently developed, integration is connected with differentiation: if f is a continuous real-valued function defined on a closed interval $[a, b]$, then, once an antiderivative F of f is known, the definite integral of f over that interval is given by

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Integrals and derivatives became the basic tools of calculus, with numerous applications in science and engineering. A rigorous mathematical definition of the integral was given by Bernhard Riemann. It is based on a limiting procedure which approximates the area of a curvilinear region by breaking the region into thin vertical slabs. Beginning in the nineteenth century, more sophisticated notions of integral began to appear, where the type of the function as well as the domain over which the integration is performed has been generalised. A line integral is defined for functions of two or three variables, and the interval of integration [a,b] is replaced by a certain curve connecting two points on the plane or in the space. In a surface integral, the curve is replaced by a piece of a surface in the three-dimensional space. Integrals of differential forms play a fundamental role in modern differential geometry. These generalizations of integral first arose from the needs of physics, and they play an important role in the formulation of many physical laws, notably those of electrodynamics. Modern concepts of integration are based on the abstract mathematical theory known as Lebesgue integration, developed by Henri Lebesgue.

The major advance in integration came in the 17th century with the independent discovery of the fundamental theorem of calculus by Newton and Leibniz. The theorem demonstrates a connection between integration and differentiation. This connection, combined with the comparative ease of differentiation, can be exploited to calculate integrals. In particular, the fundamental theorem of calculus allows one to solve a much broader class of problems. Equal in importance is the comprehensive mathematical framework that both Newton and Leibniz developed. Given the name infinitesimal calculus, it allowed for precise analysis of functions within continuous domains. This framework eventually became modern Calculus, whose notation for integrals is drawn directly from the work of Leibniz.

While Newton and Leibniz provided a systematic approach to integration, their work lacked a degree of rigor. Bishop Berkeley memorably attacked infinitesimals as "the ghosts of departed quantities". Calculus acquired a firmer footing with the development of limits and was given a suitable foundation by Cauchy in the first half of the 19th century. Integration was first rigorously formalized, using limits, by Riemann. Although all bounded piecewise continuous functions are Riemann integrable on a bounded interval, subsequently more general functions

were considered, to which Riemann's definition does not apply, and Lebesgue formulated a different definition of integral, founded in measure theory. Other definitions of integral, extending Riemann's and Lebesgue's approaches, were proposed.

If a function has an integral, it is said to be integrable. The function for which the integral is calculated is called the integrand. The region over which a function is being integrated is called the domain of integration. If the integral does not have a domain of integration, it is considered indefinite (one with a domain is considered definite). In general, the integrand may be a function of more than one variable, and the domain of integration may be an area, volume, a higher dimensional region, or even an abstract space that does not have a geometric structure in any usual sense.

The simplest case, the integral of a real-valued function f of one real variable x on the interval $[a, b]$, is denoted by

$$\int_a^b f(x)\, dx.$$

The $\int$ sign, an elongated "s", represents integration; a and b are the lower limit and upper limit of integration, defining the domain of integration; f is the integrand, to be evaluated as x varies over the interval $[a,b]$; and dx is the variable of integration. In correct mathematical typography, the dx is separated from the integrand by a space (as shown). Some authors use an upright d (that is, dx instead of dx).

The variable of integration dx has different interpretations depending on the theory being used. For example, it can be seen as strictly a notation indicating that x is a dummy variable of integration, as a reflection of the weights in the Riemann sum, a measure (in Lebesgue integration and its extensions), an infinitesimal (in non-standard analysis) or as an independent mathematical quantity: a differential form. More complicated cases may vary the notation slightly.

Integrals appear in many practical situations. Consider a swimming pool. If it is rectangular, then from its length, width, and depth we can easily determine the volume of water it can contain (to fill it), the area of its surface (to cover it), and the length of its edge (to rope it). But if it is oval with a rounded bottom, all of these quantities call for integrals. Practical approximations may

suffice for such trivial examples, but precision engineering (of any discipline) requires exact and rigorous values for these elements.

LEBESGUE INTEGRATION

In this chapter, we shall define the integral of a function on $¡^n$n, in a progressive way, with increasing order of complexity. Before we do so, we shall state some facts about Riemann integrability in measure theoretic language.

Simple Functions

Recall from the discussion on simple functions in previous chapter that the representation of simple functions is not unique. Therefore, we defined the canonical representation of a simple function which is unique. We use this canonical representation to define the integral of a simple function.

Definition: Let _ be a non-zero simple function on $¡^n$ having the canonical form

$$\phi(x) \; \sum_{i=}^{k} \; \alpha_i \chi_{E_1}$$

With disjoint measurable subject $E_1 \; ¡^n$ with $\mu(E_1) \; • \;$ and $a_i \; 0$ for all i and $a_i \; a_j$ for $i \; j. \quad i \; j.$ We define the Lebesgue integral of a simple function on $¡^n$, denoted as

$$\int_{¡^n}\phi(x)d\mu : . \phi(x) \; \sum_{i=}^{k} \; \alpha_i \chi_{E_1}$$

where μ is the Lebesgue measure on $¡^n$. Henceforth, we shall denote $\int\phi d\mu$ as $\int\phi dx$, for Lebesgue measure. Also, we define the integral of ϕ on $E \; ¡^n$ as,

$$\int_{E}\phi(x)dx := \int_{¡^n}\phi(x)\chi_{E}(x)dx$$

Note that the integral of a simple function is always finite. Though, we chose to define integral using the canonical representation, it turns out that integral of a simple function is independent of its representation.

Proof. Let $\phi = \sum_{i=1}^{k} a_i \chi_{E_i}$ be a representation of ϕ such that $E_i's$ are pairwise disjoint which is not

the canonical form, i.e., ai are not necessarily distinct and can be zero for some i. Let {bj} be the

distinct non-zero elements of $\{a_1, \ldots, a_k\}$, where $1 \le j \le k$. For a fixed j, we define $F_j = U_{\in_{j}}$,

where $I_j := \{i \mid i = j\}$, and hence $\phi(F_j) = \sum_{\in_{j}} \phi(E_i)$. Therefore, $\phi = \sum_{i=1}^{k} b_j \chi_{F_j}$ is a canonical form

of ϕ, since F_j's are pairwise disjoint. Thus,

$$\int_{\mathbb{R}^n} \phi(x)\, dx = \sum_j b_j \mu(F_j) = \sum_j b_j \sum_{i \in I_j} \mu(E_i) = \sum_{i=1}^{k} a_i \mu(E_i).$$

We now consider a representation of ϕ such that E_i are not necessarily disjoint. Let $\phi = \sum_{i=1}^{k} a_i \chi_{E_i}$

be any general representation of ϕ, $a_i \in \mathbb{R}$. Given any collection of subsets $\{E_i\}_1^k$ of $\mathbb{R}^n$, there

exists a collection of disjoint subsets $\{Fj\}_i^m$, for m $\le 2^k$, such that $U_j F_j \subseteq \mathbb{R}^n, U_{j=}^{m-} \cdot \subseteq U_i E_i$ and,

for each $i, E_i = U_{j\in_{j}}$. For each j, we define $b_j := \sum_{\in_{i}} a_i$ where $I_j := \{i \mid F_j \subseteq E_i\}$. Thus,

$\phi = \sum_{j=}^{m} b_j \chi_{F_j}$, where

F_j are pairwise disjoint. Hence, from first part of the proof,

$$\int_{\mathbb{R}^n} \phi(x)\, dx = \sum_{j=1}^{m} b_j \mu(F_j) = \sum_{j=1}^{m} \sum_{i \in I_j} a_i \mu(F_j)$$
$$= \sum_{i=1}^{k} \sum_{j \in I_i} a_i \mu(F_j) = \sum_{i=1}^{k} a_i \mu(E_i).$$

Hence the integral is independent of the choice of the representation.

Exercise 1: Show the following properties of integral of simple functions:

    (i)   (Linearity) For any two simple functions ϕ, ψ and $\phi, \psi \in$ ¡

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$$\int_{\mathbb{R}^n} (\alpha\phi + \beta\psi)\,dx = \alpha \int_{\mathbb{R}^n} \phi\,dx + \beta \int_{\mathbb{R}^n} \psi\,dx.$$

(ii)   (Additivity) For any two disjoint subsets $E, F \subset \mathbb{R}_n$ with finite measure

$$\int_{E\cup F} \phi\,dx = \int_E \phi\,dx + \int_F \phi\,dx.$$

iii) (Monotonicity) If $\phi \ge 0$ then $\int\phi \ge 0$, Consequently, if $\phi \le \psi$ then

$$\int_{\mathbb{R}^n} \phi\,dx \le \int_{\mathbb{R}^n} \psi\,dx.$$

(iv) (Triangle Inequality)We know for a simple function $\phi, |\phi|$ is also simple. Thus,

$$\left| \int_{\mathbb{R}^n} \phi\,dx \right| \le \int_{\mathbb{R}^n} |\phi|\,dx.$$

(v) If $\phi \le \psi$ a.e. then $\int\phi \le \int\psi$.

Example: An example of a Lebesgue integrable function which is not Riemann integral is the following: Consider the characteristic function $\chi_{\mathbb{Q}}$. We have already seen in Example 1.3 that this is not Riemann integrable.
But

$$\int_{\mathbb{R}^n} \chi_{\mathbb{Q}}(x)\,dx = \mu(\mathbb{Q}) = 0.$$

However, $\chi_{\mathbb{Q}} = 0$ a.e. and zero function is Riemann integrable. Thus, for $\chi_{\mathbb{Q}}$ which is Lebesgue integrable function there is a Riemann integrable functionin its equivalence class. Is this always

true? Do we always have a Riemannintegrable function in the equivalence class of a Lebesgue integrable function. The answer is a "no". Find an example!

BOUNDED FUNCTION WITH FINITE MEASURESUPPORT

Now that we have defined the notion of integral for a simple function, weintend to extend this notion to other measurable functions. At this juncture, the natural thing is to recall the fact proved in Theorem 1, which establishesthe existence of a sequence of simple functions ϕ_k converging point-wiseto a given measurable finite a.e. function f. Thus, the natural way of definingthe integral of the function f would be

$$\int_{\mathbb{R}^n} f(x)\, dx := \lim_{k \to \infty} \int_{\mathbb{R}^n} \phi_k(x)\, dx.$$

This definition may not be well-defined. For instance, the limit on the RHS may depend on the choice of the sequence of simple functions ϕ_k.

Example 1: Let $f \equiv 0$ be the zero function. By choosing $\phi_k = \chi_{\circ 0,1/k}$ which converges to f point-wise its integral is 1/k which also converging to zero. However, if we choose $\phi_k = k_{\chi}$ which converges point-wise to f, but $\int \phi_k = 1$ for all k and hence converges to 1.

But zero function is trivially a simple function with Lebesgue integralzero. Note that the situation is very similar to what happens in Riemann'snotion of integration. Therein we demand that the Riemann upper sum andRiemann lower sum coincide, for a function to be Riemann integrable. InLebesgue's situation too, we have that the integral of different sequences ofsimple functions converging to a function f may not coincide. The followingresult singles out a case when the limits of integral of simple functionscoincide for any choice.

Proposition: Let f be a measurable function finite a.e. on a set E offinite measure and let $\{\phi_k\}$ be a sequence of simple functions supported on Eand uniformly bounded by M such that $\phi_k(x) \to f(x)$ point-wise a.e. on E. Then $L := \lim_{k \to \infty} \int_E \phi\, dx$ is finite. Further, L is independent of the choiceof $\{\phi_k\}$, i.e., if $f = 0$ a.e. then L = 0.

Proof. Since $\phi_k(x) \to f(x)$ point-wise a.e. on E and $\mu(E) \leq \bullet 1$, by Egorov's theorem, for a given $\delta > 0$, there exists a measurable subset $F_\delta \subseteq E$ such that $\mu(E \setminus F_\delta) \leq \delta/(4M)$ and $\phi_k \to f$ uniformly on F_δ. Set $I_k := \int_E \phi_k$ We shall show that $\{I_k\}$ is a Cauchy sequence in R and hence converges.

Consider

$$
\begin{aligned}
|I_k - I_m| &\leq \int_E |\phi_k(x) - \phi_m(x)| \quad \text{(triangle inequality)} \\
&= \int_{F_\delta} |\phi_k(x) - \phi_m(x)| + \int_{E \setminus F_\delta} |\phi_k(x) - \phi_m(x)| \\
&< \int_{F_\delta} |\phi_k(x) - \phi_m(x)| + \frac{\delta}{2} \\
&< \mu(F_\delta)\frac{\delta}{2\mu(E)} + \frac{\delta}{2} \quad \text{for all } k, m > K \\
&\leq \delta \quad \text{for all } k, m > K \quad \text{(By monotonicity of } \mu\text{)}.
\end{aligned}
$$

Thus, $\{I_k\}$ is Cauchy sequence and converges to some L. If $f \geq 0$, repeating the above argument on I_k

$$
\begin{aligned}
|I_k| &\leq \int_E |\phi_k(x)| \quad \text{(triangle inequality)} \\
&= \int_{F_\delta} |\phi_k(x)| + \int_{E \setminus F_\delta} |\phi_k(x)| \\
&< \int_{F_\delta} |\phi_k(x)| + \frac{\delta}{4} \\
&< \mu(F_\delta)\frac{3\delta}{4\mu(E)} + \frac{\delta}{2} \quad \text{for all } k > K' \\
&\leq \delta \quad \text{for all } k > K' \quad \text{(By monotonicity of } \mu\text{)},
\end{aligned}
$$

we get L = 0.

We know from Theorem the existence of a sequence of simple functions ϕ_k converging point-wise to a given measurable finite a.e. function f. If, in addition, we assume f is bounded and supported on a finite measure set E, then the ϕ_k satisfy the hypotheses of above Proposition.

This motivates us to give the following definition.

Definition: Let f be bounded measurable function supported on a set E of finite measure. The integral of f is defined as

$$\int_E f(x)\,dx := \lim_{k \to \infty} \int_E \phi_k(x)\,dx,$$

where $\{\phi_k\}$ are uniformly bounded simple functions supported on the supportof f and converging point-wise to f. Moreover, for any measurable subset $F \subseteq E$,

$$\int_F f(x)\,dx := \int_E f(x)\chi_F(x)\,dx.$$

The way we defined our integral of a function, the interchange of limitand integral under point-wise convergence comes out as a gift.

Theorem 2: (Bounded Convergence Theorem. Let f_k be a sequence ofmeasurable functions supported on a finite measure set E such that $|f_k(x)| \leq M$ for all k and $x \in E$ and $f_k(x) \to f(x)$ point-wise a.e. on E. Then f isalso bounded and supported on E a.e. and

$$\lim_{k \to \infty} \int_E |f_k - f| = 0.$$

In particular

$$\lim_{k \to \infty} \int_E f_k = \int_E f.$$

Proof. Since f is a point-wise a.e. limit of f_k, $|f(x)| \leq M$ a.e. on E andhas support in E. By Egorov's theorem, for a given $\eta > 0$, there existsa measurable subset $F_\delta \subseteq E$ such that

$\mu(E \setminus F_\delta) \leq \delta/(4M)$ and $f_k \to f$ uniformly on F_δ. Also, choose $K \in N$, such that $f_k(x) - f(x)| \leq \delta/2\mu(E)$ for all $k \geq K$. Consider

$$
\begin{aligned}
\int_E |f_k(x) - f(x)| &\leq \int_{F_\delta} |f_k(x) - f(x)| + \int_{E \setminus F_\delta} |f_k(x) - f(x)| \\
&< \mu(F_\delta)\frac{\delta}{2\mu(E)} + \frac{\delta}{2} \quad \text{for all } k > K \\
&\leq \delta \quad \text{for all } k > K \quad \text{(By monotonicity of } \mu\text{).}
\end{aligned}
$$

Therefore, $|lim_{k \to \infty} \int_E f_k - f| \to 0$ and, by triangle inequality,

$$
\lim_{k \to \infty} \int_E f_k \to \int_E f
$$

It is now time to address the problem of Riemann integration which does not allow us to interchange point-wise limit and integral. We first observe that Riemann integration is same as Lebesgue integration for Riemann integrable functions and thus, by BCT, we have the interchange of limit and integral for Riemann integrable functions, when the limit is also Riemann integrable.

Theorem: If $f \in R([a, b])$ then f is bounded measurable and

$$
\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx,
$$

the LHS is in the sense of Riemann and RHS in the sense of Lebesgue.

Proof.

Since $f \in R([a, b])$, f is bounded, $|f(x)| \leq M$ for some M >0. Also, the support of f, being subset of [a, b], is finite. We need to check that f is measurable. Since f is Riemann integrable there exists two sequences of stepfunctions $\{\phi_k\}$ and $\{\psi_k\}$ such that

$$
\phi_1 \leq \ldots \leq \phi_k \leq \ldots \leq f \leq \psi_k \leq \ldots \leq \psi_1
$$

and

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$$\lim_k \int_a^b \phi_k = \lim_k \int_a^b \psi_k = \lim_k f.$$

Also, $|\phi_k| \leq M$ and $|\psi_k| \leq M$ for all k. Since Riemann integral is same as Lebesgue integral for step functions,

$$\int_a^b \phi_k = \int_{[a,b]} \phi_k \text{ and } \int_a^b \psi_k = \int_{[a,b]} \psi_k.$$

Let $\Phi \circ x \cdot := \lim_k \phi_k \circ x \cdot$ and $\Psi := \lim_k \psi_k \circ x \cdot$. Thus, $\Phi \leq f \leq \Psi$. Being limit of simple functions Φ and Ψ are measurable and by BCT,

$$\int_{[a,b]} \Phi = \lim_k \int_a^b \phi = \lim_k \int_a^b \psi = \int_{[a,b]} \Psi.$$

Thus, $\int_{[a,b]} \Psi \cdot \Phi \cdot = 0$. Moreover, since $\psi_k \cdot \phi_k \geq 0$, we must have $\Psi \cdot \Phi \geq 0$.

Thus, by Exercise 1, $\Psi \cdot \Phi = 0$ a.e. and hence $\Phi = f = \Psi$ a.e. Hence f is measurable. Thus,

$$\int_{[a,b]} f(x)\,dx = \lim_k \int_{[a,b]} \phi_k = \int_a^b f(x)\,dx.$$

The same statement is not true, in general, for improper Riemann integration.

NON-NEGATIVE FUNCTIONS:

We have already noted that for a general measurable function, defining its integral as the limit of the simple functions converging to it, may not be well-defined. However, we know from the proof of Theorem that any non-negative function f has truncation f_k which are each bounded and supported on a set of finite measure, increasing and converge point-wise to f. There could be many other choices of the sequences which satisfy similar condition. This motivates a definition of integrability for non-negative functions.

Definition: Let f be a non-negative $\circ f \geq 0 \cdot$ measurable function. The integral of f is defined as,

$$\int_{\mathbb{R}^n} f(x)\,dx = \sup_{0 \le g \le f} \int_{\mathbb{R}^n} g(x)\,dx$$

where g is a bounded measurable function supported on a finite measure set.

As usual,

$$\int_E f(x)\,dx = \int_{\mathbb{R}^n} f(x)\chi_E(x)\,dx$$

since $f\chi_E \ge 0$ too, if $f \ge 0$.

Note that the supremum could be infinite and hence the integral couldtake infinite value.

Definition 1: We say a non-negative function f is Lebesgue integrable

If

$$\int_{\mathbb{R}^n} f(x)\,dx < +\infty.$$

Exercise 2: Show the following properties of integral for non-negative measurable functions:

   (i)   (Linearity) For any two measurable functions f, g and $\alpha, \beta \ge$,

$$\int_{\mathbb{R}^n} (\alpha f + \beta g)\,dx = \alpha \int_{\mathbb{R}^n} f\,dx + \beta \int_{\mathbb{R}^n} g\,dx.$$

   (ii)   (Additivity) For any two disjoint subsets $E, F \subseteq \mathbb{R}^n$ with finite measure

$$\int_{E \cup F} f\,dx = \int_E f\,dx + \int_F f\,dx.$$

iii) (Monotonicity) If $f \le g$, then

$$\int_{\mathbb{R}^n} f\,dx \le \int_{\mathbb{R}^n} g\,dx.$$

In particular, if g is integrable and $0 \le f \le g$, then f is integrable

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