International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 03 Issue 14
October2016

Application of Complex Integral on Solving Some Integral Problems of Trigonometric Functions

Chii-Huei Yu
Department of Information Technology, Nan Jeon University of Science and Technology, Tainan City, Taiwan
E-mail:chiihuei@mail.nju.edu.tw

Abstract

This paper uses the mathematical software Maple for the auxiliary tool to study two types of integral problems of trigonometric functions. We can obtain the closed forms of the two types of integrals using a complex integral formula. In addition, we propose two related integrals and calculate their definite integrals practically. Our research method is to find solutions through manual calculations and verify these solutions using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking.

Key Words:

complex integral; trigonometric functions; closed forms; Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights.

In calculus and engineering mathematics, there are many methods to solve the integral problems,for example, change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. In this paper, weconsider the following twotypes of integralproblems of trigonometric functions, which are not easy to obtain their answers using the methods mentioned above.

$$\int \frac{\begin{bmatrix} -r\sin\theta\cdot[1+e^{r\cos\theta}\cos(r\sin\theta)] \\ +r\cos\theta\cdot e^{r\cos\theta}\sin(r\sin\theta) \end{bmatrix}}{1+2e^{r\cos\theta}\cos(r\sin\theta)+e^{2r\cos\theta}}d\theta, \tag{1}$$

$$\int \frac{\begin{bmatrix} r\cos\theta\cdot[1+e^{r\cos\theta}\cos(r\sin\theta)] \\ +r\sin\theta\cdot e^{r\cos\theta}\sin(r\sin\theta) \end{bmatrix}}{1+2e^{r\cos\theta}\cos(r\sin\theta)+e^{2r\cos\theta}}d\theta, \tag{2}$$

where r, θ are any real numbers.The closed formsof the twotypes of integralscan be obtainedby using a complex integral formula; these arethe major results of this article (i.e., Theorems1 and 2).Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve some integral problems. On the other hand, Yu [4-29], and Yu and Chen [30]used some methods, for example, complex power series method, integration term by term theorem, Parseval's theorem, and generalized Cauchy integral formula to solve some types of integral problems. In this study, we propose two examples to demonstrate the manual calculations, andverifythe results using Maple.

2. Preliminaries and Main Results

First, we introduce a definition and aformulaused in this paper.

2.1 Definition:

The complex logarithmic function $\ln z$ is defined by $\ln z = \ln|z| + i\theta$, where z is a complex number, $i = \sqrt{-1}$, θ is a real number, $z = |z| \cdot e^{i\theta}$, and $-\pi < \theta \le \pi$.

2.2Euler'sFormula:

$e^{i\theta} = \cos\theta + i\sin\theta$, where θ is any real number.

Toobtain the major results in this article, we need the following complex integral formula.

Lemma*Suppose that z is a complex number and C is a constant, then*

$$\int \frac{1}{1+e^z} dz = z - \ln(1+e^z) + C. \quad (3)$$

Proof $\int \frac{1}{1+e^z} dz$

$$= \int \left(1 - \frac{e^z}{1+e^z}\right) dz$$

$= z - \ln(1+e^z) + C$. q.e.d.

In the following, we determine the closed forms of the integrals (1) and (2).

Theorem 1*Let r, θ be real numbers and C_1 be a constant, then*

$$\int \frac{\begin{bmatrix} -r\sin\theta \cdot [1+e^{r\cos\theta}\cos(r\sin\theta)] \\ +r\cos\theta \cdot e^{r\cos\theta}\sin(r\sin\theta) \end{bmatrix}}{1+2e^{r\cos\theta}\cos(r\sin\theta)+e^{2r\cos\theta}} d\theta$$

$$= r\cos\theta - \ln\sqrt{1+2e^{r\cos\theta}\cos(r\sin\theta)+e^{2r\cos\theta}} + C_1. \quad (4)$$

ProofLet $z = re^{i\theta}$ in Eq. (3), then

$$\int \frac{ire^{i\theta}}{1+e^{re^{i\theta}}} d\theta = re^{i\theta} - \ln(1+e^{re^{i\theta}}) + C.$$

And hence,

$$\int \frac{ir(\cos\theta + i\sin\theta)}{1+e^{r\cos\theta}\cos(r\sin\theta)+ie^{r\cos\theta}\sin(r\sin\theta)} d\theta$$

$$= r\cos\theta + ir\sin\theta$$
$$-\ln[1+e^{r\cos\theta}\cos(r\sin\theta)+ie^{r\cos\theta}\sin(r\sin\theta)] + C.$$

Thus,

$$\int \frac{(-r\sin\theta + ir\cos\theta)\begin{bmatrix} 1+e^{r\cos\theta}\cos(r\sin\theta) \\ -ie^{r\cos\theta}\sin(r\sin\theta) \end{bmatrix}}{[1+e^{r\cos\theta}\cos(r\sin\theta)]^2 + [e^{r\cos\theta}\sin(r\sin\theta)]^2} d\theta$$

$$= r\cos\theta + ir\sin\theta$$
$$-\ln\sqrt{[1+e^{r\cos\theta}\cos(r\sin\theta)]^2 + [e^{r\cos\theta}\sin(r\sin\theta)]^2}$$
$$-i\tan^{-1}\left(\frac{e^{r\cos\theta}\sin(r\sin\theta)}{1+e^{r\cos\theta}\cos(r\sin\theta)}\right) + C.$$

(by definition)

Therefore,

$$\int \frac{(-r\sin\theta + ir\cos\theta)\begin{bmatrix}1+e^{r\cos\theta}\cos(r\sin\theta)\\ -ie^{r\cos\theta}\sin(r\sin\theta)\end{bmatrix}}{1+2e^{r\cos\theta}\cos(r\sin\theta)+e^{2r\cos\theta}}d\theta$$

$$= r\cos\theta + ir\sin\theta$$

$$-\ln\sqrt{1+2e^{r\cos\theta}\cos(r\sin\theta)+e^{2r\cos\theta}}$$

$$-i\tan^{-1}\left(\frac{e^{r\cos\theta}\sin(r\sin\theta)}{1+e^{r\cos\theta}\cos(r\sin\theta)}\right)+C. \quad (5)$$

Using the equality of the real parts of both sides of Eq. (5), we obtain Eq. (4).

q.e.d.

Theorem 2*If r,θ are real numbers and C_2 is a constant, then*

$$\int \frac{\begin{bmatrix}r\cos\theta\cdot[1+e^{r\cos\theta}\cos(r\sin\theta)]\\ +r\sin\theta\cdot e^{r\cos\theta}\sin(r\sin\theta)\end{bmatrix}}{1+2e^{r\cos\theta}\cos(r\sin\theta)+e^{2r\cos\theta}}d\theta$$

$$= r\sin\theta - \tan^{-1}\left(\frac{e^{r\cos\theta}\sin(r\sin\theta)}{1+e^{r\cos\theta}\cos(r\sin\theta)}\right)+C_2. \quad (6)$$

ProofBythe equality of the imaginary parts of both sides of Eq. (5), the desired result holds. q.e.d.

3. Examples

For the integralproblems discussed in this paper, two examples are proposed and we use Theorems 1 and 2 to determine their closed forms. Moreover, we employ Maple to calculate the approximations of some definite integrals and their solutions to verify our answers.

Example 1 Let $r = 2$ in Theorem1, we have

$$\int \frac{\begin{bmatrix}-2\sin\theta\cdot[1+e^{2\cos\theta}\cos(2\sin\theta)]\\ +2\cos\theta\cdot e^{2\cos\theta}\sin(2\sin\theta)\end{bmatrix}}{1+2e^{2\cos\theta}\cos(2\sin\theta)+e^{4\cos\theta}}d\theta$$

$$= 2\cos\theta - \ln\sqrt{1+2e^{2\cos\theta}\cos(2\sin\theta)+e^{4\cos\theta}}+C_1. \quad (7)$$

Thus the following definite integral

$$\int_0^{\pi/2} \frac{\begin{bmatrix}-2\sin\theta\cdot[1+e^{2\cos\theta}\cos(2\sin\theta)]\\ +2\cos\theta\cdot e^{2\cos\theta}\sin(2\sin\theta)\end{bmatrix}}{1+2e^{2\cos\theta}\cos(2\sin\theta)+e^{4\cos\theta}}d\theta$$

$$= -\ln\sqrt{2+2\cos 2} - 2 + \ln(e^2+1). \quad (8)$$

Next, we use Maple to verify the correctness of Eq. (8).

```
>evalf(int((-2*sin(theta)*(1+exp(2*cos(theta))*cos(2*sin(theta)))+2*cos(theta)*exp(2*cos(theta))*sin(2*sin(theta)))/(1+2*exp(2*cos(theta))*cos(2*sin(theta))+exp(4*cos(theta))),theta=0..Pi/2),15);
```

0.049407300869037

```
>evalf(-ln(sqrt(2+2*cos(2)))-2+ln(exp(2)+1),15);
```

0.049407300869041

Example 2 If $r = 3$ in Theorem2, then using Eq. (6) yields

$$\int \frac{\begin{bmatrix}3\cos\theta\cdot[1+e^{3\cos\theta}\cos(3\sin\theta)]\\ +3\sin\theta\cdot e^{3\cos\theta}\sin(3\sin\theta)\end{bmatrix}}{1+2e^{3\cos\theta}\cos(3\sin\theta)+e^{6\cos\theta}}d\theta$$

$$= 3\sin\theta - \tan^{-1}\left(\frac{e^{3\cos\theta}\sin(3\sin\theta)}{1+e^{3\cos\theta}\cos(3\sin\theta)}\right)+C_2. \quad (9)$$

Therefore, we have the following definite integral

$$\int_{\pi/6}^{\pi/2} \frac{\begin{bmatrix} 3\cos\theta \cdot [1+e^{3\cos\theta}\cos(3\sin\theta)] \\ +3\sin\theta \cdot e^{3\cos\theta}\sin(3\sin\theta) \end{bmatrix}}{1+2e^{3\cos\theta}\cos(3\sin\theta)+e^{6\cos\theta}} d\theta$$

$$= \frac{3}{2} - \tan^{-1}\left(\frac{\sin 3}{1+\cos 3}\right) + \tan^{-1}\left(\frac{e^{3\sqrt{3}/2}\sin(3/2)}{1+e^{3\sqrt{3}/2}\cos(3/2)}\right). \quad (10)$$

We also employ Maple to verify the correctness of Eq. (10).

```
>evalf(int((3*cos(theta)*(1+exp(3*cos(theta
))*cos(3*sin(theta)))+3*sin(theta)*exp(3*co
s(theta))*sin(3*sin(theta)))/(1+2*exp(3*cos(
theta))*cos(3*sin(theta))+exp(6*cos(theta)))
,theta=Pi/6..Pi/2),15);
```

1.42629228651923

```
>evalf(3/2-arctan(sin(3)/(1+cos(3)))+arctan(
exp(3*sqrt(3)/2)*sin(3/2)/(1+exp(3*sqrt(3)/
2)*cos(3/2))),15);
```

1.42629228651916

4. Conclusion

From the discussion above,we know that using complex integral can easily solve some integral problems of trigonometric functions. In fact, the applications of complex integralare extensive, and can be used to deal with many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our results.

References:

[1] A. A. Adams, H. Gottliebsen, S. A. Linton, and U. Martin, " Automated Theorem Proving in Support of Computer Algebra: Symbolic Definite Integration as a Case Study, " *Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation*, Canada, pp. 253-260, 1999.

[2] M. A. Nyblom, "On the Evaluation of a Definite Integral Involving Nested Square Root Functions, " *Rocky Mountain Journal of Mathematics*, Vol. 37, No. 4, pp. 1301-1304, 2007.

[3] C. Oster, "Limit of a Definite Integral, " *SIAM Review*, Vol. 33, No. 1, pp. 115-116, 1991.

[4] C. -H. Yu, " Solving Some Definite Integrals Using Parseval's Theorem," *American Journal of Numerical Analysis*, Vol. 2, No. 2, pp. 60-64, 2014.

[5] C. -H. Yu, " Some Types of Integral Problems, " American Journal of Systems and Software, Vol. 2, No. 1, pp. 22-26, 2014.

[6] C. -H. Yu, "Application of Parseval's Theorem on Evaluating Some Definite Integrals," *Turkish Journal of Analysis and Number Theory*, Vol. 2, No. 1, pp. 1-5, 2014.

[7] C. -H. Yu, "Evaluation of Two Types of Integrals Using Maple, " *Universal Journal of Applied Science*, Vol. 2, No. 2, pp. 39-46, 2014.

[8] C. -H. Yu, "Studying Three Types of Integrals with Maple, " *American Journal of Computing Research Repository*, Vol. 2, No. 1, pp. 19-21, 2014.

[9] C. -H. Yu, " The application of Parseval's theorem to integral problems, " *Applied Mathematics and Physics*, Vol. 2, No. 1, pp. 4-9, 2014.

[10] C. -H. Yu, "A Study of Some Integral Problems Using Maple," *Mathematics and Statistics*, Vol. 2, No. 1, pp. 1-5, 2014.

[11] C. -H. Yu, "Solving Some Definite Integrals by Using Maple," *World Journal of Computer Application and Technology*, Vol. 2, No. 3, pp. 61-65, 2014.

[12] C. -H. Yu, "Using Maple to Study Two Types of Integrals," *International Journal of Research in Computer Applications and Robotics*, Vol. 1, Issue. 4, pp. 14-22, 2013.

[13] C. -H. Yu, "Solving Some Integrals with Maple," *International Journal of Research in Aeronautical and Mechanical Engineering*, Vol. 1, Issue. 3, pp. 29-35, 2013.

[14] C. -H. Yu, "A Study on Integral Problems by Using Maple," *International Journal of Advanced Research in Computer Science and Software Engineering*, Vol. 3, Issue. 7, pp. 41-46, 2013.

[15] C. -H. Yu, "Evaluating Some Integrals with Maple," *International Journal of Computer Science and Mobile Computing*, Vol. 2, Issue. 7, pp. 66-71, 2013.

[16] C. -H. Yu, "Application of Maple on Evaluation of Definite Integrals," *Applied Mechanics and Materials*, Vols. 479-480 (2014), pp. 823-827, 2013.

[17] C. -H. Yu, "Application of Maple on the Integral Problems," *Applied Mechanics and Materials*, Vols. 479-480 (2014), pp. 849-854, 2013.

[18] C. -H. Yu, "Using Maple to Study the Integrals of Trigonometric Functions," *Proceedings of the 6th IEEE/International Conference on Advanced Infocomm Technology*, Taiwan, No. 00294, 2013.

[19] C. -H. Yu, "A Study of the Integrals of Trigonometric Functions with Maple," *Proceedings of the Institute of Industrial Engineers Asian Conference 2013*, Taiwan, Springer, Vol. 1, pp. 603-610, 2013.

[20] C. -H. Yu, "Application of Maple on the Integral Problem of Some Type of Rational Functions," (in Chinese) *Proceedings of the Annual Meeting and Academic Conference for Association of IE*, Taiwan, D357-D362, 2012.

[21] C. -H. Yu, "Application of Maple on Some Integral Problems," (in Chinese) *Proceedings of the International Conference on Safety & Security Management and Engineering Technology 2012*, Taiwan, pp. 290-294, 2012.

[22] C. -H. Yu, "Application of Maple on Some Type of Integral Problem," (in Chinese) *Proceedings of the Ubiquitous-Home Conference 2012*, Taiwan, pp.206-210, 2012.

[23] C. -H. Yu, "Application of Maple on Evaluating the Closed Forms of Two Types of Integrals," (in Chinese) *Proceedings of the 17th Mobile Computing Workshop*, Taiwan, ID16, 2012.

[24] C. -H. Yu, "Application of Maple: Taking Two Special Integral Problems as Examples," (in Chinese) *Proceedings of the 8th International Conference on Knowledge Community*, Taiwan, pp.803-811, 2012.

[25] C. -H. Yu, "Evaluating Some Types of Definite Integrals," *American Journal*

of Software Engineering, Vol. 2, Issue. 1, pp. 13-15, 2014.

[26] C. -H. Yu, "A Study of an Integral Related to the Logarithmic Function with Maple," *International Journal of Research*, Vol. 3, Issue. 1, pp. 1049-1054, 2016.

[27] C. -H. Yu, "Solving Real Integrals Using Complex Integrals," *International Journal of Research*, Vol. 3, Issue. 4, pp. 95-100, 2016.

[28] C. -H. Yu, "Integral Problems of Trigonometric Functions," *International Journal of Scientific Research in Science and Technology*, Vol. 2, Issue. 1, pp. 63-67, 2016.

[29] C. -H. Yu, "Expressions of Some Complicated Integrals," *International Journal of Scientific Research in Science and Technology*, Vol. 2, Issue. 1, pp. 59-62, 2016.

[30] C. -H. Yu and B. -H. Chen, "Solving Some Types of Integrals Using Maple," *Universal Journal of Computational Mathematics*, Vol. 2, No. 3, pp. 39-47, 2014.