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The Numerical Solution of the Fredholm Integral Equations of the Second Kind

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ABSTRACT:
In this thesis we focus on the mathematical and numerical aspects of the Fredholm integral equation of the second kind due to their wide range of physical application such as heat conducting radiation, elasticity, potential theory and electrostatics. After the classification of these integral equations we will investigate some analytical and numerical methods for solving the Fredholm integral equation of the second kind. Such analytical methods include: the degenerate kernel methods, the Adomain decomposition method, the modified decomposition method and the method of successive approximations. The numerical methods that will be presented here are: Projection methods including collocation method and Galerkin method, Degenerate kernel approximation methods and Nystrom methods. The mathematical framework of these numerical methods together with their convergence properties will be analyzed. Some numerical examples implementing these numerical methods have been obtained for solving a Fredholm integral equation of the second kind. The numerical results show a closed agreement with the exact solution.
Keywords: Fredholm integral equation, Galerkin method, Nystrom methods

INTRODUCTION:
The subject of integral equations is one of the most important mathematical tools in both pure and applied mathematics. Integral equations play a very important role in modern science such as numerous problems in engineering and mechanics, for more details. In fact, many physical problems are modeled in the form of Fredholm integral equations, such problems as potential theory and Dirichlet problems which discussed, electrostatics, mathematical problems of radiative equilibrium, the particle transport problems of astrophysics and reactor theory, and radiative heat transfer problems which discussed,. Many initial and boundary value problems associated with ordinary differential equations (ODEs) and partial differential equations (PDEs) can be solved more effectively by integral equations methods. Integral equations also form one of the most useful tools in many branches of pure analysis, such as the theories of functional analysis and stochastic processes.

Definition

An integral equation is an equation in which the unknown function φ to be determined appears under the integral sign. A standard integral equation is of the form

$$\varphi(x) = f(x) + \lambda \int_{v(x)}^{u(x)} K(x,t)\, \varphi(t)dt \quad (1.1)$$

Where $u(x)$ and $v(x)$ are limits of integration, λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t called the kernel of the integral equation. The functionφ that will be determinedappears under the integral sign, and sometimes outside the integral sign.The functions $f(x)$ and $K(x,t)$ are known and$\varphi(x)$ is the unknown function. Thelimitsofintegration $u(x)$and $v(x)$ may be both variables, constantsor mixed, and they may be inone dimension or two or more.

Fredholm integral equations
The most standard form of Fredholm integral equations is given by the form

$$\alpha(x)\varphi(x) = f(x) + \lambda \int_D K(x,t)\varphi(t)dt \quad (1.2)$$

with$D$ a closed bounded set in R^m, for some $m \geq 1$.

Case(i) If the function $\alpha(x) = 0, then$ (1.2) yields

$$f(x) + \lambda \int_D K(x,t)\, \varphi(t)\, dt \quad (1.3)$$

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The equation involved the unknown function φ only under the integral sign. In this case the integral equation is called Fredholm integral equation of the first kind.

Case(ii) If the function $\alpha(x) = 1$, then(1.2)becomes

$$\varphi(x) = f(x) + \lambda \int_D K(x,t)\ \varphi(t)dt \qquad (1.4)$$

The equation involved the unknown function φ both inside as well as outside the integral equation. In this casetheequation is called Fredholm integral equation of the second kind.

Case(iii) If $\alpha(x)$ is neither $0\ nor\ 1$ then (1.2) called Fredholm integral equation of the third kind

Volterra integral equations

The most standard form ofVolterra integral equations is given by the form

$$\alpha(x)\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\varphi(t)dt \quad (1.5)$$

where the upper limit of integration is a variable and the unknown function φ appears linearly or nonlinearly under the integral sign.

Case(i) If the function $\alpha(x) = 0$, then equation (1.5) becomes

$$f(x) + \lambda \int_D^x K(x,t)\ \varphi(t)dt \quad (1.6)$$

The equation involved the unknown function φ appearsonly under the integral sign. In this case theintegral equation is called the Volterra integral equation of the first kind.

Case(ii) If the function $\alpha(x) = 1$ then equation (1.5) becomes

$$\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\ \varphi(t)dt \qquad (1.7)$$

The equation involved the unknown function φ both inside as well as outside the integral sign. In this case the Integral Equation is called the volterra integral equation of the second kind.

Case(iii) If $\alpha(x)$ is nither 0 nor 1 then (1.5) called Volterraintegral equation of the third kind.

Singular integral equations

When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is called singular. For example,

$$\varphi(x) = f(x) + \lambda \int_{-\infty}^{\infty} (exp - |x-t|)\varphi(t)dt \quad (1.8)$$

is a singular integral equation of the second kind.

Case (i) Singular Integral Equation: The kernel is of the form

$$K(x,t) = \frac{H(x,t)}{x-t}$$

Where $H(x,t)$ is a differentiable function of (x,t) with $H(x,t) \neq 0$, then the integral equation is said to be a singular equation with Cauchy kernel where the integral

$$\int_a^b \frac{K(x,t)}{x-t}\ \varphi(t)dt$$

is understood in the sense of Cauchy Principal Value (CPV) and the notation P.V

$$\int_a^b \frac{H(x,t)}{x-t}\ dt$$

is usually used to denote this. Thus

$$\text{P.V}\int_a^b \frac{H(x,t)}{x-t}\ dt = \lim_{\epsilon\to 0}\left\{\int_a^{x-\epsilon} \frac{H(x,t)}{x-t}\ dt + \int_{x+\epsilon}^b \frac{H(x,t)}{x-t}\ dt\right\}$$

Case(ii) Weakly singular integral equation: Here the kernel is of the form

$$K(x,t) = \frac{H(x,t)}{|X-t|^{\delta}}$$

Or

$$K(x,t) = H(x,t)\ln|x-t|$$

Where $H(x,t)$ is bounded $a \le x \le b\ and\ a \le t \le b$ with $H(x,t) \neq 0, and\ \delta$ is a constant such that $0 < \delta < 1$.For example, the equation

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$$f(x) = \lambda \int_0^x \frac{1}{(x-t)^{\delta}} \varphi(t)dt, 0 < \delta < 1 \quad (1.9)$$

Is a singular integral equation with a weakly singular kernel.

Case (iii) Strongly singular integral equations : If the kernel $K(x,t)$ is of the form

$$K(x,t) = \frac{H(x,t)}{(x-t)^2}$$

Where $H(x,t)$ is a differentiable function of $(x,t)$with $(x,t) \neq 0$, then the integral equation is said to be a strongly singular integral equation. For more details see [22].

Integro Differential equations

In this type of equations, the unknown function φ appears as a combination of both ordinary derivative and under the integral sign. In the electrical engineering problem, the current $I(y)$ flowing in a closed circuit containingresistance, inductance and capacitance is governed by the following integro-differential equation,

$$L\frac{dI}{dy} + RI + \frac{I}{C}\int_0^t I(\tau)\, d\tau = E(y) \quad (1.10)$$

Where L is the inductance, R the resistance, C the capacitance, and $E(y)$ be the applied voltage. Similar examples can be cited as follows

$$\varphi'(x) = 1 - \frac{1}{3}x + \int_0^1 xt\, \varphi(t)dt, \quad \varphi(0) = 1 \quad (1.11)$$

$$\varphi''(x) = f(x) + \lambda \int_0^x (x-t)\varphi(t)\, dt \quad , \quad \varphi(0) = 0, \varphi'(0) = 1 \quad (1.12)$$

SOLVING FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND:

The existence and uniqueness

Some integral equation has a solution and some other has no solution or that it has an infinite number of solutions, the following theorems state the existence and uniqueness among the solution of Fredholm integral equation of the second kind.

Note: It is important to say that we will discuss the analytical methods in the space

$$X = [a,b]\ with\ \|.\|_{\infty}.$$

Theorem (Fredholm alternative)

Either the nonhomogeneous linear equation of second kind

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\, \varphi(t)dt \quad (2.1)$$

has a unique solution for any function $f(x)$ (in some sufficiently broad class) or the corresponding homogeneous equation

$$\varphi(x) = \lambda \int_a^b K(x,t)\, \varphi t\, dt \quad (2.2)$$

has at least one nontrivial (that is, not identically zero) solution.

Theorem. If the first alternative holds true for equation (2.1), then it holds true for the associated equation

$$\psi(x) = g(x) + \lambda \int_a^b K(t,x)\, \psi(t)\, dt \quad (2.3)$$

as well. The homogeneous integral equation (2.2) and its associated equation

$$\psi(x) = \lambda \int_a^b K(t,x)\, \psi(t)\, dt \quad (2.4)$$

have one and the same finite number of linearly independent solutions.

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Note: If the functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ are solutions of the homogeneous equation (2.2), then their linear combination

$$\varphi(x) = C_1\varphi_1(x) + C_2\varphi_2(x) + \cdots + C_n\varphi_n(x) = \sum_{i=1}^{n} C_i\varphi_i(x) \qquad (2.5)$$

where the $C_i (i = 1,2, \dots, n)$ are arbitrary constants, is also a solution of the equation.

Theorem. A necessary and sufficient condition for the existence of a solution $\varphi(x)$ of the non-homogeneous equation (2.1) in the latter case of the alternative is the condition of a orthogonality of the right side of the equation, i.e., of the function $f(x)$, to any solution $\psi(x)$ of the homogeneous equation (2.4) associated with (2.2)

$$\int_a^b f(x)\psi(x)\, dx$$

Some Analytical Methods for solving Fredholm integral equations of the second kind

The degenerate kernel method

The kernel $K(x,t)$ of a Fredholm integral equation of the second kind is called degenerate if it is the sum of a finite number of products of functions of x alone by functions of t alone, i.e.., if it is of the form

$$K(x,t) = \sum_{i=1}^{n} u_i(x)v_i(t) \qquad (2.6)$$

We shall consider the functions $u_i(x)$ and $v_i(t)$, $(i = 1,2,3, \dots, n)$ continuous in the basic square $a \le x, t \le b$ and linearly independent. The integral equation with degenerate kernel (2.6).

$$\varphi(x) = f(x) + \lambda \int_a^b \left[\sum_{i=1}^{n} u_i(x)v_i(t)\right]\varphi(t)\, dt \qquad (2.7)$$

is solved in the following manner .

Rewrite (2.7) as,

$$\varphi(x) = f(x) + \lambda \sum_{i=1}^{n} u_i(x) \int_a^b v_i(t)\, \varphi(t)\, dt \qquad (2.8)$$

and introduce the notation

$$\int_a^b u_i(t)\, \varphi(t)\, dt = C_i (i = 1,2, \dots, n) \qquad (2.9)$$

Then (2.8) becomes

$$\varphi(x) = f(x) + \lambda \textstyle\sum_{i=1}^{n} C u_i(x) \qquad (2.10)$$

Where C_i are unknown constants, since the function $\varphi(x)$ is unknown. Thus, the solution of an integral equation with degenerate kernel reduces to finding the constants $C_i (i = 1,2, \dots, n)$. Putting the expression (2.10) into the integral equation (2.7), we get

$$\textstyle\sum_{m=1}^{n}\{C_m - \int_a^b v_m(t)\,[f(t) + \lambda \sum_{i=1}^{n} C_i u_i(t)]dt\}u_m(x) = 0$$

Whence it follows, by virtue of the linear independence of the functions $u_m(x) (m = 1,2, \dots, n)$, that

$$C_m - \textstyle\int_a^b v_m(t)[f(t) + \lambda \sum_{i=1}^{n} C_i u_i(t)]dt = 0$$

Or

$$C_m - \lambda \textstyle\sum_{i=1}^{n} C_i \int_a^b u_i(t)\ v_m(t)dt = \int_a^b v_m(t)f(t)dt, \quad (m = 1,2, \dots, n)$$

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For the sake of brevity, we introduce the notations

$$a_{im} = \int_a^b u_i(t)v_m(t)dt,$$

$$f_m = \int_a^b v_m(t)f(t)dt$$

And find that

$$C_m - \lambda\sum_{i=1}^{n} u_{im}C_i = f_m \quad , \quad (m = 1,2,\dots,n) \quad (2.11)$$

Or, in expanded form

$$\begin{vmatrix} (1-\lambda u_{11})C_1 - \lambda u_{12}C_2 \dots\dots \lambda u_{1n}C_n = f_1 \\ -\lambda u_{21}C_1 + (1-\lambda u_{22})C_2 \dots\dots \lambda u_{2n}C_n = f_2 \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \cdot \\ \cdot \\ -\lambda u_{n1}C_1 - \lambda u_{n2}C_2 \dots\dots + (1-\lambda u_{nn})C_n = f_n \end{vmatrix} \quad (2.12)$$

For finding the unknowns c_i, we have a linear system of n algebraic equation in n unknowns. The determinant of this system is

$$D(\lambda) = \begin{vmatrix} 1-\lambda u_{11} & -\lambda u_{12} \dots & -\lambda u_{1n} \\ -\lambda u_{21} & 1-\lambda u_{22} \dots & -\lambda u_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -\lambda u_{n1} & -\lambda u_{n2} \dots & 1-\lambda u_{nn} \end{vmatrix} \quad (2.13)$$

For all values of λ for which $D(\lambda) \neq 0$, the algebraic system (2.11), and there by the integral equation (2.7), has a unique solution. On the other hand, for all values of λ for which $D(\lambda)$ becomes equal to zero, the algebraic system (2.11), and with it the integral equation (2.7), either is insoluble or has an infinite number of solutions. Note that we have considered only the integral equation of the second kind, where alone this method is applicable.

Example solve the integral equation

$$\varphi(x) - \lambda\int_{-\pi}^{\pi}(x\cos t + t^2\sin x + \cos x \sin t)\,\varphi(t)dt = x \quad (2.14)$$

Now, write the equation (2.14) in the following from,

$$\varphi(x) = \lambda x\int_{-\pi}^{\pi}\varphi(t)\cos t\,dt + \lambda\sin x\int_{-\pi}^{\pi}t^2\,\varphi(t)dt + \lambda\cos x\int_{-\pi}^{\pi}\varphi(t)\sin t\,dt + x$$

We introduce the notation

$$C_1 = \int_{-\pi}^{\pi}\varphi(t)\cos t\,dt\,; \quad C_2 = \int_{-\pi}^{\pi}t^2\varphi(t)dt \quad ; \quad C_3 = \int_{-\pi}^{\pi}\varphi(t)\sin t\,dt \quad (2.15)$$

Where C_1, C_2, C_3 are unknown constants. Then equation (2.14) assumes the form

$$\varphi(x) = C_1\,\lambda x + C_2\lambda\sin x + C_3\lambda\cos x + x \quad (2.16)$$

Substituting expression (2.16) into (2.15), we get

$$C_1 = \int_{-\pi}^{\pi}(C_1\lambda t + C_2\lambda\sin t + C_3\lambda\cos t + t)\cos t\,dt\,,$$

$$C_2 = \int_{-\pi}^{\pi}(C_1\lambda t + C_2\lambda\sin t + C_3\lambda\cos t + t)\,t^2\,dt\,,$$

$$C_3 = \int_{-\pi}^{\pi}(C_1\lambda t + C_2\lambda\sin t + C_3\lambda\cos t + t)\sin t\,dt$$

Or

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$$C_1\left(1-\lambda\int_{-\pi}^{\pi} t\cos t\,dt\right) - C_2\lambda\int_{-\pi}^{\pi}\sin t\cos t\,dt - C_3\lambda\int_{-\pi}^{\pi}\cos^2 t\,dt = \int_{-\pi}^{\pi} t\cos t\,dt$$

$$-C_1\lambda\int_{-\pi}^{\pi} t^3\,dt - C_2\left(1-\lambda\int_{-\pi}^{\pi} t^2\sin t\,dt\right) - C_3\lambda\int_{-\pi}^{\pi} t^2\cos t\,dt =$$

$$\int_{-\pi}^{\pi} t^3\,dt - C_1\lambda\int_{-\pi}^{\pi} t\sin t\,dt - C_2\lambda\int_{-\pi}^{\pi}\sin^2 t\,dt + C_3\left(1-\lambda\int_{-\pi}^{\pi}\cos t\sin t\,dt\right) = \int_{-\pi}^{\pi} t\,sint\,dt$$

By evaluating the integrals that enter into this system we obtain a system of algebraic equations for finding the unknowns C_1, C_2, C_3.

$$\left.\begin{aligned} C_1 - \lambda\pi C_3 &= 0 \\ C_2 + 4\lambda\pi C_3 &= 0 \\ -2\lambda\pi C_1 - \lambda\pi C_2 + C_3 &= 2\pi \end{aligned}\right\} \quad (2.17)$$

The determinant of this system is

$$D(\lambda) = \begin{vmatrix} 1 & 0 & -\pi\lambda \\ 0 & 1 & 4\pi\lambda \\ -2\pi\lambda & -2\pi & 1 \end{vmatrix} = 1 + 2\lambda^2\pi^2 \neq 0$$

The system (2.17) has a unique solution

$$C_1 = \frac{2\pi^2\lambda}{1+2\lambda^2\pi^2}\ ,\quad C_2 = \frac{8\pi^2\lambda}{1+2\lambda^2\pi^2}\ ,\quad C_3 = \frac{2\pi}{1+2\lambda^2\pi^2}$$

Substituting the values of C_1, C_2, C_3 thus found into (2.16) we obtain the solution of the given integral equation

$$\varphi(x) = \frac{2\lambda\pi}{1+2\lambda^2\pi^2}(2\pi x + 4\lambda\pi\,sin\,x + cosx) + x$$

For more examples see [54]

The Method of successive approximations : Neumann's series

The successive approximation method, which was successfully applied to Volterra integral equations of the second kind, can be applied even more easily to the basic Fredholm integral equations of the second kind:

$$\varphi(x) = f(x) + \lambda\int_a^b K(x,t)\varphi(t)\,dt \quad (2.18)$$

We set $\varphi_0(x) = f(x)$. Note that the zeroth approximation can be any selected real-valued function $\varphi_0(x)$, $a \le x \le b$. Accordingly, the first approximation $\varphi_1(x)$ of the solution of $\varphi(x)$ is defined by

$$\varphi_1(x) = f(x) + \lambda\int_a^b K(x,t)\varphi_0(t)\,dt \quad (2.19)$$

The second approximation $\varphi_2(x)$ of the solution $\varphi(x)$ can be obtained by replacing $\varphi_0(x)$ in equation (2.19) by the previously obtained $\varphi_1(x)$; hence we find

$$\varphi_2(x) = f(x) + \lambda\int_a^b K(x,t)\varphi_1(t)\,dt \quad (2.20)$$

This process can be continued in the same manner to obtain the nth approximation. In other words, the various approximation can be put in a recursive scheme given by

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$\varphi_0(x) =$ any selective real valued function

$$\varphi_n(x) = f(x) + \lambda \int_a^b K(x,t)\varphi_{n-1}^{(t)}\, dt, \quad n \geq 1 \quad (2.21)$$

Even though we can select any real-valued function for the zeroth approximation $\varphi_0(x)$, the most commonly selected functions for $\varphi_0(x)$ are $\varphi_0(x) = 0, 1, or\ x$. We have noticed that with the selection of $\varphi_0(x) = 0$,the first approximation$\varphi_1(x) = f(x)$. The final solution $\varphi(x)$ is obtained by

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) \quad (2.22)$$

So that the resulting $\varphi(x)$ is independent of the choice of$\varphi_0(x)$. This is known as Picard's method.

The Neumann series is obtained if we set $\varphi_0(x) = f(x)$ such that

$$\varphi_1(x) = f(x) + \lambda \int_a^b K(x,t)\varphi_0(t)\, dt$$

$$= f(x) + \lambda \int_a^b K(x,t) f(t) dt$$

$$= f(x) + \lambda\psi_1(x) \quad (2.23)$$

Where

$$\psi_1(x) = \int_a^b K(x,t) f(t) dt \quad (2.24)$$

The second approximation $\varphi_2(x)$ can be obtained as

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x,t)\varphi_1(t)\, dt$$

$$= f(x)\lambda \int_a^b K(x,t)\{f(t) + \lambda\, \psi_1(t)\} dt$$

$$= f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) \quad (2.25)$$

Where

$$\psi_2(x) = \int_a^b K(x,t)\psi_1(t)\, dt \quad (2.26)$$

Proceeding in this manner , the final solution $\varphi(x)$ can be obtained

$$u(x) = f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + \cdots + \lambda^n\psi_n(x) + \cdots$$

$$= f(x)\sum_{n=1}^{\infty} \lambda^n\psi_n(x)$$

Where

$$\psi_n(x) = \int_a^b K(x,t)\psi_{n-1}(t)\, dt \quad (2.28)$$

Example

Solve the Fredholm Integral equation

$$\varphi(x) = 1 + \int_0^1 x\varphi(t)\, dt$$

By using the successive approximation method.

For solution let us consider the zeroth approximation is $\varphi_0(x) = 1$, and then the first approximation can be computed as

$$\varphi_1(x) = 1 + \int_0^1 x\varphi_0(t)\, dt$$

$$= 1 + \int_0^1 x\, dt$$

$$= 1 + x$$

Proceeding in this manner,we find

$$\varphi_2(x) = 1 + \int_0^1 x\varphi_1(t)\, dt$$

$$= 1 + \int_0^1 x(1+t)\, dt$$

$$= 1 + x\left(1 + \frac{1}{2}\right)$$

Similarly, the third approximation is

$$\varphi_3(x) = 1 + x\int_0^1 \left(1 + \frac{3t}{2}\right) dt$$

$$= 1 + x\left(1 + \frac{1}{2} + \frac{1}{4}\right)$$

Thus, we get

$$\varphi_n(x) = 1 + x\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\dots\dots + \frac{1}{2^{n-1}}\right\}$$

And hence

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$$

$$= 1 + \lim_{n\to\infty} x \sum_{d=0}^{n} \frac{1}{2^d}$$

$$= 1 + x\left(1 - \frac{1}{2}\right)^{-1}$$

$$= 1 + 2x$$

This is the desired solution.

Numerical Methods for Solving Fredholm Integral Equations of the Second Kind

Degenerate kernel approximation methods

We discussed the degenerate kernel method as an analytical method in chapter two (2.3.1) for solving the Fredholm integral equation

$$\varphi(x) = f(x) + \lambda \int_D K(x,t)\, \varphi(t)\, dt\ ,\ x \in D \qquad (3.1)$$

with $\lambda \neq 0$ and $D \subset R^m$, for some $m \geq 1$, where D is a closed and bounded set.

We said that the kernel $K(x,t)$ is degenerate if it can be expressed as the sum of a finitenumber of terms, each of which is the product of a function of x only and a function of t only such that

$$K(x,t) = \sum_{i=1}^{n} u_i(x)v_i(t) \qquad (3.2)$$

but most kernel functions $K(x,t)$ are not degenerate. So that in this chapter we seek to approximate them by degenerate kernels.

The solution of the integral equation by the degenerate kernel method

In the view of the integral equation (3.1), the kernel function $K(x,t)$ is to be approximated by a sequence of degenerate kernel functions,

$$K_n(x,t) = \sum_{i=1}^{n} u_{i,n}(x)v_{i,n}(t), \qquad n \geq 1 \qquad (3.3)$$

In such a way that the associated integral operators G_n satisfy

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$\lim_{n\to\infty} \| G - G_n \| = 0$(3.4)

Where the associated integral operator is defined as

$G_n\varphi(x) = \int_D K_n(x,t)\varphi(t)dt, \quad x \in D, \varphi \in C(D), \quad n \geq 1$ (3.5)

Where D is a closed bounded set in R^m, for some $n \geq 1$, and using $X = C(D)$ with $\|.\|_\infty$, such that $G: C(D) \to C(D)$ is compact.

We can write the integral equation (3.1) in the operator from as

$(1 - \lambda G)\varphi = f$ (3.6)

Then (3.6) can be written using (3.5) as

$(1 - \lambda G_n)\varphi_n = f$ (3.7)

Where φ_n is the solution of the approximating equation. Using the formula (3.3) for $K_n(x,t)$, the integral equation (3.7) becomes

$\varphi_n(x) = f(x) + \lambda \sum_{i=1}^{n} u_{i,n}(x) \int_D v_{i,n}(t)\varphi_n(t)dt$ (3.8)

And using the technique discussed in section (2.31) we have
$\varphi_n(x) = f(x) + \lambda \sum_{i=1}^{n} a_i u_i(x)$ (3.9)

Where

$a_i - \lambda \sum_{d=1}^{n} C_{id}\, a_d = h_i \qquad i = 1, \dots, n$ (3.10)

Such that

$h_i = \int v_i(y)g(t)dt$ (3.11)

And

$c_{id} = \int v_i(y)u_d(t)dt$ (3.12)

Are known constants. Again as we stated in section (2.31) equation (3.10) represents a system of n algebraic equations for the unknowns α_i whose determinant $D(\lambda)$ is given by

$$D(\lambda) = \begin{vmatrix} 1-\lambda C_{11} & -\lambda C_{12} \dots & -\lambda C_{1n} \\ -\lambda C_{21} & 1-\lambda C_{22} \dots & -\lambda C_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ -\lambda C_{n1} & -\lambda C_{n2} \dots & 1-\lambda C_{nn} \end{vmatrix}$$ (3.13)

Which is a polynomial in λ of degree at most n,that is not identically zero.

To analyze the solution of (3.1) by the degenerate kernel method the following situations arise:

Situation I: when at least one right member of the system (3.9) $h_1, h_2, \dots h_n$ is non zero, the following two cases arise under this situation.

(i) If $D(\lambda) \neq 0$, then a unique non zero solution of system (3.10) exists and so (3.1) has unique non zero solution given by (3.9).

(ii) If $D(\lambda) = 0$, then the system (3.10) have either no solution or they possess infinite solution and hence (3.1) has neither no solution or infinite solution.

Situation II:When $f(x) = 0$, then (3.11) shows that $h_i = 0$ for $i = 1,2, \dots, n$. Hence the system (3.10) reduces to a system of homogeneous linear equation. The following two cases arises under this situation.

(i) If $D(\lambda) \neq 0$, then a unique non zero solution $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ of the system (3.10) exists and so we see that (3.1) has unique zero solution $\varphi_n(x) = 0$.

(ii) If $D(\lambda) = 0$, then the system (3.10) posses infinite non zero solutions and so (3.1) has infinite

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non zero solutions, those value of λ for which $D(\lambda) = 0$ are known as eigenvalues and any nonzero solution of the homogeneous Fredholm integral equation $\varphi(x) = \int_D K(x,t)\,\varphi(t)\,dt$ is known as a corresponding eigenfunction of integral equation.

Situation III: When $f(x) \neq 0$ but

$\int_D f(t)v_1(t) = 0\,, \int_D f(t)v_2(t) = 0\,, \dots\,,$
$\int_D f(t)v_n(t) = 0$ (3.14)

that is $f(x)$ is orthogonal to all the functions

$v_1(t), v_2(t) \dots, v_n(t)$ (3.15)

Then

$h_1, h_2, \dots h_n$ are zeros and reduces (3.11) to a system of homogeneous linear equations. The following two cases arise under this situation

(i) If $D(\lambda) \neq 0$, then a unique zero solution $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, and hence (3.1) has only unique solution $\varphi_n(x) = 0$.

(ii) If $D(\lambda) = 0$, then the system (3.10) possess infinite nonzero solutions and hence (3.1) has infinite nonzero solutions.

Taylor series approximation

Let $\varphi(x,t)$ is a continuous function of two variables $x$and t, then the Taylor series expansion of function φ at the neighborhood of any real number a with respect to the variable y is :

$Taylor\,(\varphi, t, a) = \sum_{n=0}^{\infty} \frac{(t-a)^n}{n!} \frac{\partial^n}{\partial t^n} \varphi(x, a)$ (3.16)

and

$Taylor\,(\varphi, t, m, a) = \sum_{n=0}^{m} \frac{(t-a)^n}{n!} \frac{\partial^n}{\partial t^n} \varphi(x, t = a)$ (3.17)

that mean the *mth* terms of Taylor expansion to the function at the neighborhood a with respect to the variable t.

Consider the one-dimensional integral equation

$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\,\varphi(t)\,dt$ (3.18)

Often we can write K as a power series in s,
$K(x,t) = \sum_{i=0}^{\infty} K_i(x)(t-a)^i$ (3.19)
or in x
$K(x,t) = \sum_{i=0}^{\infty} K_i(t)(x-a)^i$ (3.20)
Let K_n denote the partial sum of the first *n* terms on the right side of (3.19)
$K_n(x,t) = \sum_{i=0}^{n-1} K_i(x)(t-a)^i$ (3.21)
Using the notation of (3.3), K_n is a degenerate kernel with
$u_i = K_{i-1}(x)\ ,\ \ v_i(t) = (t-a)^{i-1}\ ,\ \ i = 1,2, \dots, n$ (3.22)
The linear system (3.14) becomes
$\alpha_i - \lambda \sum_{j=1}^{n} \alpha_j \int_a^b (t-a)^{i-1} K_{j-1}(t)dt =$
$\int_a^b f(t)(t-a)^{i-1}dt, \quad i = 1, \dots, n$ (3.23)
and the solution φ_n is given by
$\varphi_n(x) = f(x) + \lambda \sum_{i=0}^{n-1} \alpha_{i+1} K_i(x)$(3.24)

The integrals in (3.23) must often be calculated numerically. However, there is not much that can be said for integrals of this generality. First, they involve the entire interval [a, b], as contrasted with some later methods we consider. In addition, most of the integrands will be zero or quite small, in the neighborhood of $t = a$, the left end of the interval. The latter may aid in choosing a more efficient method of numerical integration. The following example avoids the numerical calculation of most of these integrals.

Interpolator degenerate kernel approximations

Interpolation is a simple way to obtain degenerate kernel approximations. There are many kinds of interpolation, but we consider interpolation using only the values of $K(x,t)$.There are many candidates for interpolation functions, including polynomials,

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trigonometric polynomials, piecewise polynomial functions (including spline functions), and others. We give a general framework for all of these,

Let $\psi_1(x) \dots \psi_n(x)$ be a basis for the space of interpolation functions we are using. For example, with polynomial interpolation of degree $< n$, we would use

$\psi_i(x) = x^{i-1} \quad , \quad 1 \le i \le n$ (3.25)

Let $x_1 \dots x_n$ be interpolation nodes in the integration region *D*. The *interpolation problem* is as follows: Given data $t_1 \dots t_n$find

$z(x) = \sum_{j=1}^{n} \alpha_j \psi_j(x)$(3.26)

With

$z(x_i) = t_i, \quad i = 1, \dots, n$ (3.27)

Thus, we want to find the coefficients $\alpha_1, \dots, \alpha_n$ solving the linear system

$\sum_{j=1}^{n} \alpha_j \psi_j(x_i) = t_i \quad i = 1, \dots, n$ (3.28)

In order for the interpolation problem to have a unique solution for all possible data $\{t_i\}$, it is necessary and sufficient that

$\det(\Gamma_n) \neq 0, \quad \Gamma_n = [\psi_j(x_i)]$ (3.29)

With polynomial interpolation and the basis of (3.25)

$$\Gamma_n = \left[x_i^{\,j-1}\right]_{i,j=1}^{n}$$

This is called a *Vandermonde* matrix, and it is known that $\det(\Gamma_n) \neq 0$ for all distinct choices of $x_1 \dots x_n$.

To give an explicit formula for $K_n(x,t)$we introduce a special basis for the interpolation method. Define $\ell_d(t)$to be the interpolation function for which

$$\ell_d(t_i) = \delta_{id}, \quad i = 1, \dots, n$$

Then the solution to the interpolation problem is given by

$z(x) = \sum_{j=1}^{n} t_j \ell_j(x)$ (3.30)

For polynomial interpolation, this is called Lagrange's form of the interpolation polynomial. We often use this name when dealing with other types of interpolation, and the functions $\ell_d(t)$ are usually called Lagrange basis functions. With polynomial interpolation,

$$\ell_d(x) = \prod_{\substack{i=1 \\ i \neq d}}^{n} \left(\frac{x - x_i}{x_d - x_i}\right)$$

Interpolation with respect to the variable t

Define

$K_n(x,t) = \sum_{j=1}^{n} \underbrace{\ell_j(x)}_{u_j(x)} \underbrace{K(x_j,t)}_{v_j(t)}$(3.31)

Then $K_n(x_i,t) = K(x_i,t), \ i = 1, \dots, n$all $t \in D$. For the case D = [a, b],

With $K(x,t)$ being considered on the domain $[a,b] \times [a,b]$, we have that $K_n(x_i,t)\ equals\ K(x,t)$alongall lines$x = x_i$.

The linear system $A_n\alpha = r$associated with the degenerate kernel method $(1 - \lambda G_n)\varphi_n = f$ is

$\alpha_i - \lambda \sum_{j=1}^{n} \alpha_j \int_D \ell_j(t) K(x_i,t)dt = \int_D K(x_i,t) f(t)\, dt, \quad i = 1, \dots, n$ (3.32)

The solution, φ_n is given by,

$\varphi_n(x) = f(x) + \lambda \sum_{j=1}^{n} \alpha_j \ell_j(x)$ (3.33)

Note the integrals in (3.32) must generally be evaluated numerically.

When analyzing this degenerate kernel method within the context of the space $C(D)$, the error depends on

$\|G - G_n\| = \max_{x \in D} \int_D |K(x,t) - K_n(x,t)|\, dt$ (3.34)

which in turn depends on the interpolation error$K(X,t) - K_n(X,t)$. Some special cases are considered below.

NUMERICAL EXAMPLES AND RESULTS

In this chapter we try to apply some of the numerical methods illustrated in chapter three to approximate the solution of the Fredholm integral equation

$\varphi(x) = -\frac{2}{\pi}\cos(x) + \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x - t)\, \varphi(t)dt$ (4.1)

These methods include: **the degenerate kernel method, the Nyström methodandthe collocation method** we will use suitable algorithms and Matlab software,

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then we will compare the exact solution with the approximate one using suitable number of n points.

The numerical realization of equation (4.1) using the degenerate kernel method

First we expand the kernel $K(x,t)$ withrespect to t using the Taylor seriessuch that

$$Taylor\ (K,t,a) = \sum_{n=0}^{m} \frac{(t-a)^n}{n!} \frac{\partial^n}{\partial t^n} K(x,t=a) \quad (4.2)$$

where m is the number of Taylor series terms, by this expansion, the kernel can be written as the sum of two separated functions one with respect to x, and the other with respect to t, such that

$$K_m(x,t) = \sum_{i=0}^{m-1} u_i(x) v_i(t) \quad (4.3)$$

Where

$$u_{i-1}(x) = \left(\frac{1}{i!}\right) \frac{\partial^{i-1}}{\partial t^{i-1}} K(x,a) \quad (4.4)$$

And

$$v_{i-1}(t) = (t-a)^{i-1}, \forall i = 1,2,\dots m \quad (4.5)$$

then we calculate the values c_{ij}, and h_i such that

$$c_{ij} = \int_a^b v_i(y) u_j(t)\, dt\ ,$$

$$h_i = \int_a^b v_i(t)\, g(t) dt \quad i = 1,2,\dots,m \quad (4.6)$$

using the relations in section 2.3.1, and the above relations, we have

$$\alpha_i - \lambda \sum_{j=1}^{n} c_{ij} \alpha_j = h_i, \quad i = 1,\dots,n \quad (4.7)$$

now putting this relations in the matrix form we have, where

$$A[\alpha_i] = H$$

Where

$$A = I - \lambda C$$

such that I is the identity matrix,

$$C = [c_{ij}], \forall\ i,j = 1,2,\dots,m$$

$$H = [h_i], \quad \forall\ i = 1,2,\dots,m$$

And the matrix

$$[\alpha_i] = A^{-1} H$$

The solution φ_m is given by

$$\varphi_m(x) = f(x) + \lambda \sum_{i=0}^{m-1} \alpha_{i+1} f_i(x) \quad (4.8)$$

The following algorithm implements the **degenerate kernel method** using the **Matlab**software.

Algorithm 1

1. Input a, b, $\lambda, f(x), K(x,t)$
2. Input the number of Taylor series terms m
3. Calculate the Taylor expansion of $K(x,t)$ with respect to t from φ find $u_i(x)\ and\ v_i(t), i = 1,2,\dots,m$
4. Calculate $c_{id} = \int_a^b v_i(t) u_d(t) dt \quad i,d = 1,2,\dots,m$
5. Calculate $h_i = \int_a^b v_i(t)\, f(t) dt \quad i = 1,2,\dots,m$
6. Calculate the matrix

$$A = \begin{vmatrix} 1-\lambda c_{11} & -\lambda c_{12} \dots & -\lambda c_{1m} \\ -\lambda c_{21} & 1-\lambda c_{22} \dots & -\lambda c_{2m} \\ . & . & . \\ . & . & . \\ . & . & . \\ -\lambda c_{m1} & -\lambda c_{m2} \dots & 1-\lambda c_{mm} \end{vmatrix}$$

7. Calculate the determinate D(A) of matrix A
8. If $f(x) \neq 0$go to step 12.
9. If $D(A) = 0$ the system has infinite number of solutions ,go to step 16
10. The system has unique solution $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$go to step 16
11. If $h_i = 0$go to step 15
12. If D(A) = 0, the system has infinite number of solutions, go to step 16, thesystem has unique solution $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$
13. If $D(A) = 0$,the system has no real solution, go to step 16
14. The solution of system is $[\alpha_i] = [A_{id}]^{-1}[h_i]^T$ then

$$\varphi_m(x) = f(x) + \lambda \sum_{i=1}^{m} \alpha_i u_i\,(x)$$

1. End.

For more details see [20].

By returning to**3.1.2** and using algorithm 1, the kernel of this integral equation $K(x,t) = \cos(x-t)$can be expanded using Taylor series for 5 terms as

$$Taylor(\cos(x-t)\,, t, 5) = \cos(x) + t\sin(x) - \frac{t^2}{2}\cos(x) - \frac{t^3}{2}\sin(x) + \frac{t^4}{24}\cos(x) \quad (4.10)$$

Implies

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$$\left.\begin{aligned} u_1(x) &= \cos(x) \\ u_2(x) &= \sin(x) \\ u_3(x) &= -\tfrac{1}{2}\cos(x) \\ u_4(x) &= -\tfrac{1}{6}\sin(x) \\ u_5(x) &= \tfrac{1}{24}cos(x) \end{aligned}\right\} \qquad \left.\begin{aligned} v_1(t) &= 1 \\ v_2(t) &= t \\ v_3(t) &= t^2 \\ v_4(t) &= t^3 \\ v_5(t) &= t^4 \end{aligned}\right\}$$

(4.11) (4.12)

and

The related Matlab program gives the following results.

The matrix C =

1.0000	1.0000	-0.5000	-0.1667	0.0417
0.5708	1.0000	-0.2854	-0.1667	0.0238
0.4674	1.1416	-0.2337	-0.1903	0.0195
0.4510	1.4022	-0.2255	-0.2337	-0.1895
0.4793	1.8040	-0.2396	-0.3007	0.0200

The Matrix $A = I - \lambda C =$

-0.2732	-1.2732	0.6366	0.2122	-0.0531
-0.7268	-0.2732	0.3634	0.2122	-0.0303
-0.5951	-1.4535	1.2976	0.2423	-0.0248
-0.5742	-1.7853	0.2871	1.2976	0.2413
-0.6102	-2.2970	0.3051	0.3828	0.9746

The matrix $[\alpha_i] = A^{-1}H =$

0.8752
0.9251
1.0775
0.8782
1.7330

then

$$\varphi_m = \frac{-2}{\pi}\cos(x_j) + \frac{4}{\pi}[\alpha_i][u_i(x_j)], \quad i,j = 1,2,\dots,m \qquad (4.13)$$

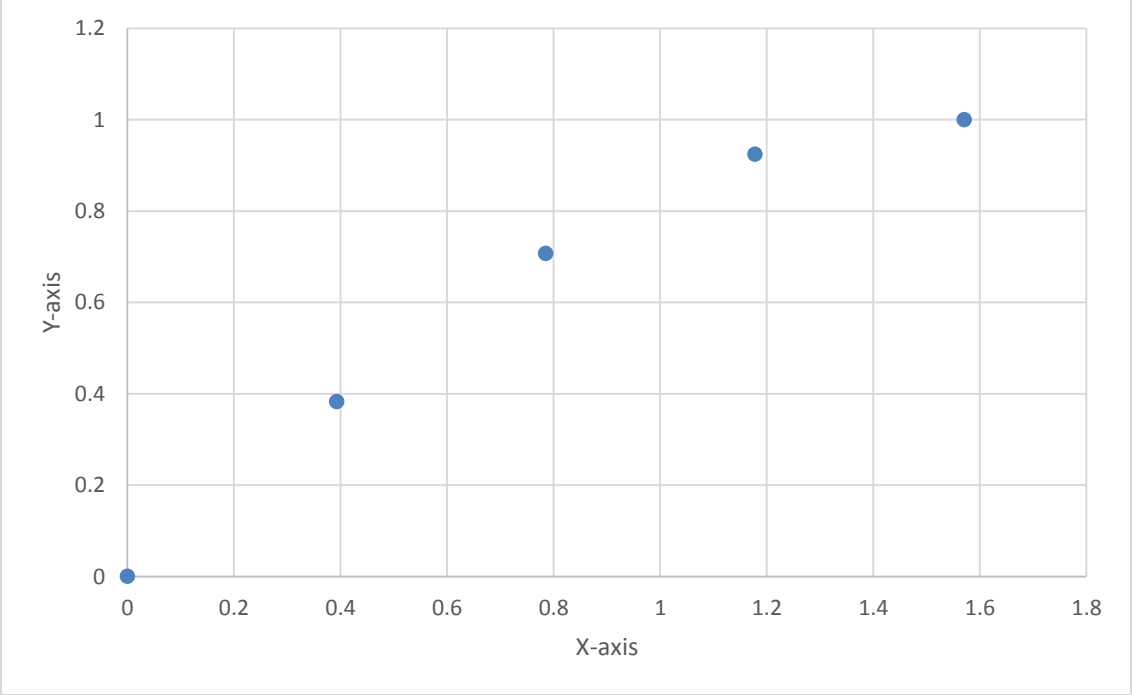
where

$$x_{j+1} = x_j + \frac{(b-a)}{m-1}, \quad and \ x_1 = a \qquad (4.14)$$

Table 1: The exact and numerical solution of applying Algorithm 1 for equation (4.1).

x	Analytical solution $t_1 = \sin(x)$	Approximate solution t_2	Error = $\lvert t_1 - t_2 \rvert$
0	0	-0.116299822082018	0.116299822082018
0.3927	0.382683432365090	0.271988984127792	0.110694448237297
0.7854	0.707106781186547	0.618869933090427	0.088236848096121
1.1781	0.923879532511287	0.871533544809957	0.052345987701329
1.5708	1.000000000000000	0.991514074803429	0.008485925196571

Figure 2: shows the exact solution $\varphi(x) = sin\,(x)$ and the approximate one when$m = 5$



The exact and numerical solution of applying Algorithm 1 for equation (4.1). While Figure shows the absolute error which approaches zero.

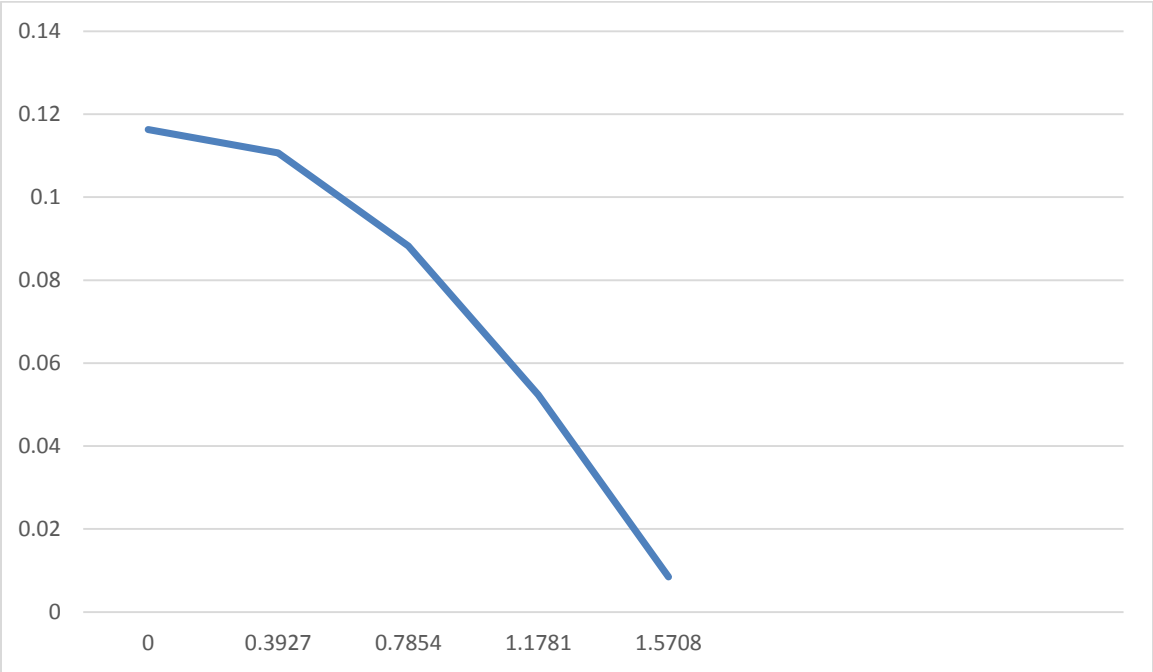


Figure 3: The resulting error of applying algorithm 1 to equation (4.1).

The numerical realization of equation (4.1) using the Nyström method

To solve the Fredholm integral equation of the second kind which is given by

$$\varphi(x) = -\frac{2}{\pi}\cos(x) + \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x-t)\,\varphi(t)dt$$

by**Nyström method**, first we should remember that the kernel $cos(x-t)$ and the function $-\frac{2}{\pi}\cos(x)$ must be continuous, secondly, we should know that we can approximate the integral $\int_a^b \psi(t)dt$ using quadrature rule by $\sum_{j=0}^{n} w_j \psi(t_j)$. By such approximation, for $a \le x \le b$, the Fredholm integral equation

$$\varphi(x) = f(x) + \lambda \int_D K(x,t)\,\varphi(t)dt\,,\ x \in D \qquad (4.15)$$

can be reduced to

$$\varphi_n(x) = \lambda \sum_{j=1}^{n} w_j\, K(x\,, x_j)\varphi_n(x_j) + f(x) \qquad (4.16)$$

where its solution $\varphi_n(x)$ is an approximation of the exact solution $\varphi(x)$ to (4.15). A solution to a functional equation (4.16) can be obtained if we assign $x_i's$ to x in which $i = 1,2,\dots,n$and $a \le x_i \le b$. In this way, (4.16) is reduced to a system of equations

$$\varphi_n(x_i) = \lambda \sum_{j=1}^{n} w_j\, K(x_i\,, x_j)\varphi_n(x_j) + f(x_i) \qquad (4.17)$$

Next, writing the equation (4.17) in the matrix form

$$F = \lambda GDF + K$$

$$K = F - \lambda GDF$$

$$K = (I - \lambda GD)F \qquad (4.18)$$

where

$$F = [\varphi_n(x_i)]^t,\ \ K = [f(x_i)]^t,\quad G = \left[K(x_i, x_j)\right]$$

$$D = diag(w_1, w_2, \dots w_n)$$

It's worth to mention that in order to approximate the integral, we will use the Trapezoidal Rule.

Here, we implement it in the form such that

$$\int_a^b K(x,t)\ dt = \sum_{j=1}^{n} w_j\ K(x_i, x_j) = DG \tag{4.19}$$

where D is a diagonal matrix such that the elements of its diagonal equal h where h depends on the initial and the end points of the interval [a, b], and the number of the approximations n such that $h = \frac{b-a}{n}$ .The elements of the matrix G consist of the entries $G(x_i, x_j)$ where $i, j = 1,2,\dots,n$, such that the approximations $x_i's$ obtained as $x_i = a + h * i$, where $i = 2,3,\dots,n$, and $x_1 = a$.

The following algorithm implements the **Nyström method**using the **Matlab**software.

Algorithm 2

Input $a, b, n, \lambda, f(x),\ K(x)$

$$h \to \frac{b-a}{n}$$

$$x_1 = a, \qquad x_n = b$$

For $i = 2\ to\ n-1$

$$x_i = a + h * i$$

End

For $i = 1\ to\ n$

$$K_i = f(x_i)$$

$$s_i = x_i$$

$D_{ii} = h \to D$is diagonal matrix

For $i = 1\ to\ n$

$$G_{ij} = G(x_i, x_j)$$

end

$I \to$identity matrix

$$lhs \to I - \lambda DG$$

$F \to$the answer of $lhs * \varphi = K$

$p(\varphi) \to$the interpolating polynomial at $[S_i, \varphi_i]$

Table 4.2 shows the exact solution $\varphi(x) = \sin(x)$ and the approximate one when $n = 50$, and showing the error resulting of using the numerical solution.

Note: The table shows the first 10 values and the last 10 values only

Table 4.2 The exact and numerical solution of applying Algorithm 3 for equation (4.1).

x	Analytical solution $t_1 = \sin(x)$	Approximate solution t_2	Error $= \lvert t_1 - t_2 \rvert$
0	0	0.031405592470328	0.031405592470328
0.0314	0.031410759078128	0.062780191412531	0.031369432334402
0.0628	0.062790519529313	0.094092833885359	0.031302314356046
0.0942	0.094108313318514	0.125312618091103	0.031204304772588
0.1257	0.125333233564304	0.156408733871965	0.031075500307661
0.1571	0.156434465040231	0.187350493115954	0.030916028075723
0.1885	0.187381314585725	0.218107360042338	0.030726045456613

0.2199	0.218143241396543	0.248648981336784	0.030505739940241
0.2513	0.248689887164855	0.278945216106394	0.030255328941540
0.2827	0.278991106039229	0.308966165625180	0.029975059585951
1.2881	0.960293685676943	0.968423843447016	0.008130157770073
1.3195	0.968583161128631	0.975756237987680	0.007173076859049
1.3509	0.975916761938747	0.982125678925927	0.006208916987179
1.3823	0.982287250728689	0.987525880392547	0.005238629663858
1.4137	0.987688340595138	0.991951513040665	0.004263172445527
1.4451	0.992114701314478	0.995398209305166	0.003283507990688
1.4765	0.995561964603080	0.997862567712965	0.002300603109885
1.5080	0.998026728428272	0.999342156239842	0.001315427811571
1.5394	0.999506560365732	0.999835514710546	0.000328954344814

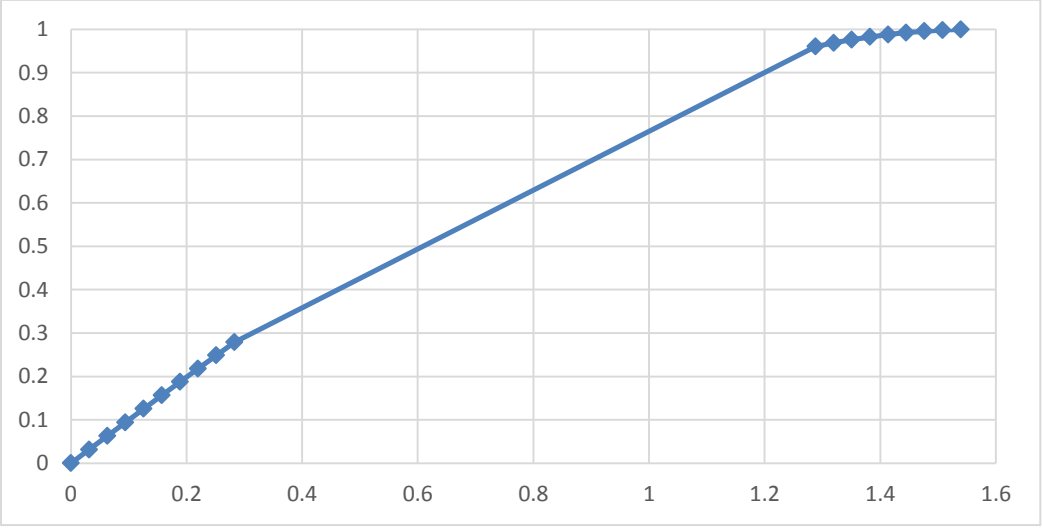


Figure 4: The exact and numerical solution of applying Algorithm 2 for equation (4.1).

The CPU time is 0.064010 seconds.

The error analysis of the Nyström method

If we consider the trapezoidal numerical integration rule

$$\int_a^b \varphi(t)dt \approx h\sum_{i=0}^{n}{}''\varphi(x_i) \quad (4.20)$$

with $h = \frac{b-a}{n}$ and $x_i = a + ih$ for $i = 0, \dots, n$. The notation Σ'' means the first and last terms are to be halved before summing. For the error,

$$\int_a^b \varphi(t)dt - h\sum_{i=0}^{n}{}''\varphi(x_i) = -\frac{h^2(b-a)}{12}\varphi''(\varepsilon_n), \quad \varphi \in [a,b], n \geq 1 \quad (4.21)$$

with ε_n some point in $[a,b]$. There is also the asymptotic error formula

$$\int_a^b \varphi(t)dt - h\sum_{i=0}^{n}{}''\varphi(x_i) = -\frac{h^2}{12}[\varphi'(b) - \varphi'(a)] + O(h^4), \quad \varphi \in C^4[a,b] \quad (4.22)$$

When this is applied to the integral equation

$$\varphi(x) = f(x) + \lambda\int_a^b K(x,t)\varphi(t)dt \quad , \quad a \leq x \leq b \quad (4.23)$$

we obtain the approximating linear system

$$\varphi_n(x_i) = f(x_i) + \lambda h\sum_{j=0}^{n}{}''K(x_i,x_j)\varphi_n(x_j), \quad i = 0,1,\dots,n \quad (4.24)$$

which is of order $q_n = n + 1$

The Nystrom interpolation formula is given by

$$\varphi_n(x) = f(x) + \lambda h\sum_{j=0}^{n}{}''K(x,x_j)\varphi_n(x_j), \quad a \leq x \leq b \quad (4.25)$$

The speed of convergence is based on the numerical integration error

$$(G - G_n)\varphi(t) = -\frac{h^2(b-a)}{12}\left[\frac{\partial^2 K(x,t)\varphi(t)}{\partial t^2}\right] \quad (4.26)$$

with $\varepsilon_n(x) \in [a,b]$. From (4.22), the asymptotic integration error is

$$(G - G_n)\varphi(f) = -\frac{h^2}{12}\left[\frac{\partial K(x,t)\varphi(t)}{\partial t}\right] + O(h^4) \quad (4.27)$$

From (4.26), we see the Nyström method converges with an order of $O(h^2)$, provided $K(x,t)\varphi(t)$ is twice continuously differentiable with respect to t, uniformly in x. For more details see [18].

These results show that the algorithm 2, yields acceptable results since the maximum absolute error which is $0.0003 \leq O(h^2)$.

The numerical realization of equation (4.1) using the Collocation method

First we expand the function $\varphi_n(x)$ as a sum of basis $\{\psi_1, \dots, \psi_n\}$ such that

$$\varphi_n(x) = \sum_{j=0}^{d_n} c_j\varphi_j(x), \quad x \in \left[0, \frac{\pi}{2}\right] \quad (4.28)$$

Since the residual $r_n(x)$ can be written as

$$r_n(x) = \varphi_n(x) - \lambda\int_D K(x,t)\varphi_n(t)dt - f(x) \quad (4.29)$$

Then by substituting (4.15) into the equation (4.16) so as to determine the values of the coefficients $\{c_1, \dots, c_d\}$, such that

$$\varphi_n(x) = \sum_{j=1}^{d} c_j\{\psi_j(x) - \lambda\int_D K(x,t)\psi_j(t)dt\} - f(x) \quad (4.30)$$

But we pick distinct node points $x_1, \dots, x_n \in D$, such that

$$r_n(x_i) = 0, \quad i = 1, \dots, n \quad (4.31)$$

Then (4.30) can be rewritten as

$$\sum_{j=1}^{d} c_j\{\varphi_j(x) - \lambda\int_D K(x,t)\varphi_j(t)dt\} = f(x) \quad (4.32)$$

In this example we have $D = [a,b], h = (b - a)/n$. Hence we take the node points are ,

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$$x_i = a + ih\,, \qquad i = 0,1,\dots,n$$

We introduce the language basis functions for piecewise linear interpolation as

$$\ell_i(x) = \begin{cases} 1 - \left|\frac{x-x_i}{h}\right|, x_{i-1} \le x \le x_{i+1} \\ 0, \qquad otherwise \end{cases} \quad (4.33)$$

Where the subspace X_n is the set of all functions that are piecewise linear on [a,b], with breakpoints $\{x_0,\dots,x_n\}$. Its dimension is $n+1$.

The projection operator is defined by

$$P_n\varphi(x) = \sum_{i=0}^{n} \varphi(x_i)\ell_i(x) \quad (4.34)$$

Now for convergence of $P_n(x)$

$$\| \varphi - P_n\varphi \|_\infty \le \begin{cases} \omega(g,h), & \varphi \in C[a,b] \\ \frac{h^2}{8} \| \varphi" \|_\infty\,, & \varphi \in C^2[a,b] \end{cases} \quad (4.35)$$

Where the function ω is defined by

$$w(\varphi,h) = \sup_{\substack{a\le x-t\le b \\ |x-t|\le h}} |\varphi(x) - \varphi(t)| \quad (4.36)$$

And it is called the modulus of the function φ.This shows that$P_n\varphi \to \varphi$ for all $\varphi \in C[a,b]$.Now for any compact operator $G: C[a,b] \to C[a,b]$ Lemma (3.6) implies $\|G - P_nG\| \to 0$ as $n \to \infty$ .Therefore the results of Theorem (3.4) can be applied directly to the numerical solution of the integral equation $(\lambda - G)\varphi = f$ For sufficiently large n, say $n \ge N$, the equation $(\lambda - P_nG)\varphi_n = P_nf$ has a unique solution φ_n for each $f \in C[a,b]$; and we can write

$$\|\varphi - \varphi_n\|_\infty \le |\lambda| M \|\varphi - P_n\varphi\|_\infty$$

For $\varphi \in C^2[a,b]$

$$\|\varphi - \varphi_n\|_\infty \le |\lambda| M \frac{h^2}{8} \|\varphi"\|_\infty \quad (4.37)$$

The linear system (4.32) takes the simpler form

$$\varphi_n(x_i) - \lambda \sum_{j=0}^{n} \varphi_n(x_j) \int_a^b K(x_i,t)\ell_j(t)dt = f(x_i),\ i = 0,\dots,n \quad (4.38)$$

And we simplify the integral for $j = 1,\dots,n-1$,

$$\int_a^b K(x_i,t)\ell_j(t)dt = \frac{1}{h}\int_{x_{j-1}}^{x_j} K(x_i,t)\big(t - x_{j-1}\big)dt + \frac{1}{h}\int_{x_j}^{x_{j+1}} K(x_i,t)\big(x_j - t\big)dt \quad (4.39)$$

We have calculated the integrals above numerically using quadrature rules specifically Trapezoidal Rule which is of the form

$$\int_a^b \varphi(x)\,dx \approx \frac{b-a}{n}\left[\frac{1}{2}\varphi(x_0) + \sum_{i=1}^{n-1}\varphi(x_i) + \frac{1}{2}\varphi(x_n)\right] \quad (4.40)$$

Now substituting (4.39) in (4.38) and putting this relation in the matrix form we have

$$F - \frac{\lambda}{h}F(GDU + GDV) = K \to \left(I - \frac{\lambda}{h}(GDU + GDV)\right)F = K \quad (4.41)$$

Where

$$F = [\varphi_n(x_i)]^T\,, \quad K = [g(x_i)]^T \quad G = \big[K\big(x_i,x_j\big)\big], D = diag(w_1,w_2,\dots,w_n)$$

$$U = \big[x_i - x_{j-1}\big] \quad , \quad V = [x_j - x_1]$$

The following algorithm implements the collocation method using the Matlab Software.

Algorithm 3

Input $a, b, n, \lambda, f(x), K(x)$

$$h \to \frac{b-a}{n}$$

$$x_1 = a, \qquad x_{n+1} = b$$

For $i = 2\ to\ n$

$$x_i = a + h * i$$

End

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For $i = 1\ to\ n+1$

$$K_i = f(x_i)$$

$$S_i = x_i$$

$D_{ii} = h \rightarrow D$is a diagonal matrix

For $j = 1\ to\ n+1$

$$G_{ij} = G(x_i, x_j)$$

end

$I \rightarrow$identity matrix

For $i = 1\ to\ n+1$

For $j = 2\ to\ n+1$

$U_{i1} = x_i - x_{j-1}$

$V_{i1} = x_j - x_1$

$U_{ij} = x_i - x_{j-1}$

$V_{ij} = x_j - x_i$

$lhs \rightarrow I - \frac{\lambda}{h}(DGU + DGV)$

$F \rightarrow the\ answer\ of\ lhs * \varphi = K$

$p(\varphi)$
$\rightarrow the\ interpolating\ polynomial\ at\ [S_i, \varphi_i]$

Table 4.2 compare the exact solution $\varphi(x) = \sin(x)$ with the approximate one when $n = 50$and showing the error resulting of using the numerical solution.

Table 3: The exact and numerical solution of applying Algorithm 3 for equation (4.1).

x	Analytical solution $t_1 = \sin(x)$	Approximate ***solution*** t_2	Error = $\lvert t_1 - t_2 \rvert$
0	0	-0.031467686762045	0.031467686762045
0.0314	0.031410759078128	-0.000000000000004	0.031410759078132
0.0628	0.062790519529313	0.031467686762042	0.031322832767271
0.0942	0.094108313318514	0.062904318716399	0.031203994602115
0.1257	0.125333233564304	0.094278871702702	0.031054361861602
0.1571	0.156434465040231	0.125560382825064	0.030874082215167
0.1885	0.187381314585725	0.156717981008673	0.030663333577051
0.2199	0.218143241396543	0.187720917465807	0.030422323930735
0.2513	0.248689887164855	0.218538596041232	0.030151291123623
0.2827	0.278991106039229	0.249140603406845	0.029850502632384
1.2881	0.960293685676943	0.952780175523255	0.007513510153688

1.3195	0.968583161128631	0.962034086005045	0.006549075123586
1.3509	0.975916761938747	0.970338584991732	0.005578176947016
1.3823	0.982287250728689	0.977685476945429	0.004601773783260
1.4137	0.987688340595138	0.984067511370779	0.003620829224359
1.4451	0.992114701314478	0.989478389970310	0.002636311344168
1.4765	0.995561964603080	0.993912772860129	0.001649191742951
1.5080	0.998026728428272	0.997366283839667	0.000660444588605
1.5394	0.999506560365732	0.999835514710550	0.000328954344819
1.5708	1.000000000000000	1.001318028640015	0.001318028640015

Figure 4: shows the exact solution $\varphi(x) = \sin(x)$ with the approximate one when $n = 50$,

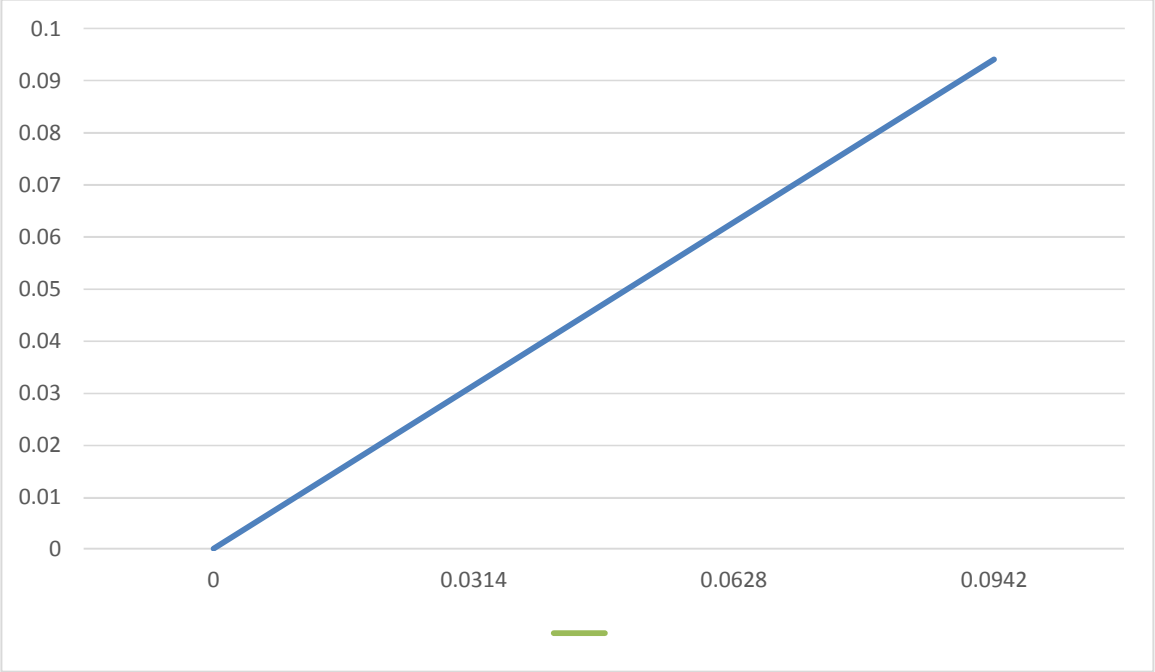

Figure 4: The exact and numerical solution of applying Algorithm 3 for equation (4.1).

The CPU time is 0.066202 seconds.

These results show that the algorithm yields acceptable results since the maximum absolute error which is 0.03 is less than or equal $O(h)$.

Conclusion:

In this thesis we have presented some analytical and numerical methods for solving a fredholm integral equation of the second kind.

The analytical methods are the degenerate kernel methods,the Adomain decomposition method, the modified decomposition method and the method of successive approximations. Moreover, we have used the following numerical methods: Projection methods including collocation method and Galerkin method, Degenerate kernel approximation methods and Nyström methods, for approximating the solution of the Fredholm integral equations. The have presented each numerical method as algorithm and applied these algorithms on the same Freedholm integral equation using Matlab Software; we have found that the numerical solution was approximately as the exact solution. The absolute error has approached zero which was shown that numerical results were acceptable.

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