Combinatorics

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Abstract—

Combinatorics is the branch of discrete mathematics studying the finite discrete structures which include enumeration, combination, and permutation of sets of elements and the mathematical relations. The term "combinatorics" refers to a larger subset of discrete mathematics that includes graph theory, which also has numerous natural connections to other areas. Combinatorics is used mostly in computer science to obtain formulas and formulate in the analysis of algorithms. Combinatorial problems arise in many areas of pure mathematics, notably in algebra, probability theory, topology, and geometry.

Definition:

Defining combinatorics within the larger field of mathematics is not an easy task. However, rather than the object of study, what characterizes combinatorics is its methods: counting arguments, induction, inclusion-exclusion, the probabilistic method - in general, surprising applications of relatively elementary tools, rather than gradual development of a sophisticated machinery. That is what makes combinatorics very elegant and accessible, and why combinatorial methods should be in the toolbox of any mainstream mathematician.

Typically, combinatorics deals with finite structures such as graphs, hyper graphs, partitions or partially ordered sets.

Keywords:

Extremal Graph Theory; Ramsey Theory; Set Theory; Combinatorial Number Theory; Discrete Geometry; Random Graphs; The Probabilistic Method

Introduction:

Combinatorial arise in many areas of professional mathematics, especially in algebra, probability theory, topology, and geometry, and it also has many applications in mathematical optimization, computer science and statistical physics. Many combinatorial questions have historically been considered in isolation, giving an ad hoc solution to a problem arising in some mathematical context. In the later twentieth century, however, powerful and general theoretical methods were founded, making combinatorics into an independent branch of mathematics in its own right. One of the ancient and most accessible parts of combinatorics is graph theory, which also has numerous natural connections to other areas. Combinatorics is used mostly in computer science to obtain formulas and estimates in the analysis of algorithms. Combinatorics is often described briefly as being about counting, and indeed counting is a large part of combinatorics. As the name suggests, however, it is broader than this: it is about combining things. Questions that arise include counting problems: "How many ways can these elements are combined?" But there are other questions, such as whether a certain combination is possible, or what combination is the "best" in some sense. We will see all of these, though counting plays a particularly large role. Combinatorics is, in essence, the study of arrangements: pairings and groupings, rankings and orderings, selections and allocations. There are three principal branches in the subject. Enumerative combinatorics is the science of counting. Problems in this subject deal with determining the number of possible arrangements of a set of objects under some particular constraints. Existential combinatorics studies problems concerning the existence of arrangements that possess some specified property. Constructive combinatorics is the design and study of algorithms for creating arrangements with special properties.

Subfields of combinatorics:

1. Enumerative combinatorics

Enumeration, otherwise known as counting, is the oldest mathematical subject, while algebraic com- binatorics is one of the youngest. Some cynics claim that algebraic combinatorics is not really a new subject but just a new name given to enumera- tive combinatorics in order to enhance its (former) poor image.

The basic parts of Enumerative combinatorics-

1.1 Union

Given two combinatorial families, $\mathcal{F}$ and $\mathcal{G}$ with generating functions $F(x)$ and $G(x)$ respectively, the union of the two families ($\mathcal{F} \cup \mathcal{G}$) has generating function $F(x) + G(x)$.

1.2 Pairs

For two combinatorial families as above the Cartesian product (pair) of the two families ($\mathcal{F} \times \mathcal{G}$) has generating function $F(x)G(x)$.

1.3 Sequences

A sequence generalizes the idea of the pair as defined above. Sequences are arbitrary Cartesian products of a combinatorial object with itself. Formally:

$$\mathrm{Seq}(\mathcal{F}) = \epsilon \cup \mathcal{F} \cup \mathcal{F} \times \mathcal{F} \cup \mathcal{F} \times \mathcal{F} \times \mathcal{F} \cup \cdots$$

To put the above in words: An empty sequence or a sequence of one element or a sequence of two elements or a sequence of three elements, etc. The generating function would be:

$$1 + F(x) + [F(x)]^2 + [F(x)]^3 + \cdots = \frac{1}{1 - F(x)}$$

2. Analytical combintorics

Analytic Combinatorics teaches a calculus that enables precise quantitative predictions of large combinatorial structures. This course introduces the symbolic method to derive functional relations among ordinary, exponential, and multivariate generating functions, and methods in complex analysis for deriving accurate asymptotic from the GF equations.

3. Extremal Combinatorics

Extremal Combinatorics is one of the central areas in Discrete Mathematics. It deals with problems that are often motivated by questions arising in other areas, including Theoretical Computer Science, Geometry and Game Theory. This paper contains a collection of problems and results in the area, including solutions or partial solutions to open problems suggested by various researchers. The topics considered here include questions in Extremal Graph Theory, Polyhedral Combinatorics and Probabilistic Combinatorics. This is not meant to be a comprehensive survey of the area, it is merely a collection of various extremal problems, which are hopefully interesting.

4. Probabilistic combinatorics

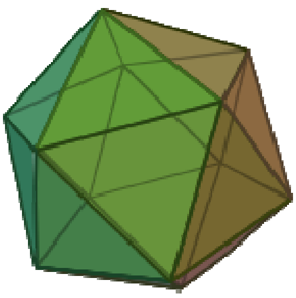
The method, which is now called the probabilistic method, is a very powerful tool for proving results in Discrete Mathematics. The early results combined combinatorial arguments with fairly elementary probabilistic techniques, whereas the development of the method in recent years required the application of more sophisticated tools from probability. The book [1] is a recent text dealing with the subject. The applications of probabilistic techniques in Discrete Mathematics, initiated by Paul Erd˝os who contributed to the development of the method more than anyone else, can be classified into three groups. The first one deals with the study of certain classes of random combinatorial objects, like random graphs or random matrices. The results here are essentially results in Probability Theory, although most of them are motivated by prob- lems in Combinatorics.

5. Algebraic combinatorics

Algebraic combinatorics involves the use of techniques from algebra, topology, and geometry in the solution of combinatorial problems, or the use of combinatorial methods to attack problems in these areas. Problems amenable to the methods of algebraic combinatorics arise in these or other areas of mathematics or from diverse parts of applied mathematics. Because of this interplay with many fields of mathematics, algebraic combinatorics is an area in which a wide variety of ideas and methods come together.

6. Geometric combinatorics

Geometric combinatorics refers to a growing body of mathematics concerned with counting properties of geometric objects described by a finite set of building blocks. Primary examples include polytopes (which are bounded polyhedral and the convex hulls of finite sets of points) and complexes built up from them. Other examples include arrangements and intersections of convex sets and other geometric objects. As we'll see there are interesting connections to linear algebra, discrete mathematics, analysis, and topology, and there are many exciting applications to economics, game theory, and biology.



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A few examples where combinatorial ideas play a key role:

1. Ramsey theory

In the 1950's, a Hungarian sociologist S. Szalai studied friendship relation- ships between children. He observed that in any group of around 20 children, he was able to find four children who were mutual friends or four children such that no two of them were friends. Before drawing any sociological conclusions, Szalai consulted three eminent mathematicians in Hungary at that time. A brief discussion revealed that indeed this is a mathematical phenomenon rather than a sociological one. For any symmetric relation R on at least 18 elements, there is a subset S of 4 elements such that R contains either all pairs in S or none of them. This fact is a special case of Ramsey's theorem proved in 1930, the foundation of Ramsey theory which developed later into a rich area of combinatorics.

2. Tournament paradox.

Suppose that n basketball teams compete in a tournament where each pair of teams plays exactly one game. The organizers want to award the k best teams. However, whichever k teams they pick, there is always another team that beats them all! Is this possible? It can be proved using a random construction that for any $k > 0$ there is $n > k$ such that this can indeed happen.

3. Brouwer's Theorem.

In 1911, Luitzen Brower published his famous Fixed Point Theorem: Every continuous map $f : Bn \rightarrow Bn$ (where Bn is an n-dimensional ball) has a fixed point, $f(x) = x$. The special case of $n = 1$ follows easily from the intermediate value theorem. For higher dimensions, however, the origianal proof was complicated. In 1928, Emanuel Sperner found a simple combinatorial result which implies Brouwer's fixed point theorem in an elegant way. The proof of Sperner's lemma is equally elegant, by double counting.

4. Borsuk's conjecture.

In 1933, Karol Borsuk published a paper which contained a proof of a conjecture .Every continuous map $f : Sn \rightarrow Rn$ (where Sn is an n-dimensional sphere) maps two antipodal points to the same value, $f(x) = f(-x)$.

Algorithmic aspects and future challenges:

The rapid development of theoretical Computer Science and its tight connection to Discrete Mathematics motivated the study of the algorithmic aspects of combinatorial results. The study of the algorithmic problems corresponding to probabilistic proofs is related to the investigation of randomized algorithms, a topic which has been developed tremendously during the last decade. In particular, it is interesting to find explicit constructions of combinatorial structures whose existence is proved by probabilistic arguments. "Explicit" here means that there is an efficient algorithm that constructs the desired structure in time polynomial in its size. Constructions of this type, besides being interesting in their own, have applications in other areas. Thus, for example, explicit constructions of error correcting codes that are as good as the random ones are of major interest in coding and information theory, and explicit constructions of certain Ramsey type colourings may have applications in de-randomization – the process of converting randomized algorithms into deterministic ones. The application of other advanced tools such as algebraic and analytic techniques, spectral methods and topological proofs, also tend to lead in many cases to non-constructive proofs. The con- version of these to algorithmic ones may well be one of the main future challenges of the area. Another interesting recent development is the in- creased appearance of computer aided proofs in Combinatorics, starting with the proof of the Four Color Theorem.

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