

Analysis of Algebra of Complex Analysis

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ABSTRACT:

The review analysis the complex analysis in algebraic geometry one reviews complex analytic and algebraic varieties, maps between such spaces (the easiest case being holomorphic and algebraic capacities) and analytic and algebraic items characterized on those spaces, as subvarieties, vector groups and bundles. There are numerous relations of complex analysis and algebraic geometry to different fields of science, for instance utilitarian analysis, algebraic topology and commutative variable based math. An established use of complex analysis is analytic number hypothesis. As of late elliptic bends, a most loved subject of study in complex analysis and algebraic geometry, have turned into an essential apparatus in algorithmic number hypothesis and in cryptography. Different parts of complex analysis and algebraic geometry (e.g. misshapening hypothesis and the hypothesis of moduli spaces) have turned out to be important for hypothetical material science.

INTRODUCTION:

A ring is a set A with two operations (called addition and multiplication) that behave like those of the integers in the precise sense follows:

A with addition is a commutative group multiplication is associative, distributive with respect to the addition, and it has a neutral element. In more detail, a ring A is a group wherein two laws are data internal composition, denoted by $+$ and \cdot satisfying the following properties:

Whatever the elements a , b and c belonging to the group A :

$$(a + b) + c = a + (b + c)$$

$$a + b = b + a$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(b + c) \cdot a = b \cdot a + c \cdot a$$

There is an element, denoted 0 and called neutral element of the internal composition law $+$, such that for any element belonging to the set A :

$$a + 0 = 0 + a = a$$

Any element belonging to the set A has an opposite, denoted $-a$, which checks:

$$a + (-a) = (-a) + a = 0$$

There is an element, denoted 1 and called neutral element of the internal composition law \cdot or unite element, such that for any element belonging to the set A :

$$a \cdot 1 = 1 \cdot a = a$$

A commutative ring is a ring whose multiplication is also commutative.

In explaining as above, this is a ring in which the following identity is verified regardless of the elements a and b of the set A :

$$a \cdot b = b \cdot a$$

Examples of commutative rings

- (i) All single-element $\{0\}$ provided operations $0 + 0 = 0$ and $0 \cdot 0 = 0$ is a commutative ring, called zero ring, or ring trivial.
- (ii) The set of integers, \mathbb{Z} with addition and multiplication is a commutative ring.
- (iii) A commutative body is a commutative ring in which all non-zero elements are invertible for multiplication.
- (iv) Among many others, the set of rational numbers, \mathbb{Q} , the set of real numbers \mathbb{R} , the set of complex numbers, \mathbb{C} , with addition and multiplication are commutative usual body, thus commutative rings.
- (v) The set of congruence classes modulo a positive integer n is a given commutative ring for the law from the congruence; it is noted $\mathbb{Z}/n\mathbb{Z}$.
- (vi) The set of polynomials with coefficients in a commutative ring is also a commutative ring.

Examples of non-commutative rings

- (i) The endomorphism of a vector space form a ring where the first law is the addition function for the law $+$ and the second composition. It is not commutative in general.
- (ii) Hamilton quaternions are a non-commutative body.
- (iii) The set \mathbb{N} of natural numbers is not a ring, because it is not a group when we equip with the addition: the existence of opposites is lacking. It is a semi-ring. The whole set \mathbb{Z} (relative) peers is not a ring, for its multiplication has no neutral element. It is a pseudo-ring.
- (iv) Allostomoms is not a ring, for its multiplication is not associative. Some times called non-associative ring.
- (v) For any non-trivial group $(G, +)$, the group $(GG+)$ G applications in G becomes, when it provides with the \circ composition, an almost-ring (in), but not even a ring g is commutative because the distributive left is not true:

We do not have $f \circ (g + h) = (f \circ g) + (f \circ h)$.

Homomorphism of rings.

A ring homomorphism is a map f between two rings A and B is consistent with their structure, the following precise sense:

For all a, b in A :

$$f(a + b) = f(a) + f(b)$$

$$f(a \cdot b) = f(a) \cdot f(b)$$

In particular, if A and B are unitary, this morphism is said unitary if

$$f(1_A) = 1_B.$$

The following applications are examples of morphisms of rings: The combination of the ring of complex

numbers to himself. This morphism is bijective (it is said that this is an auto morphism of C); For positive integer n , Z projection on the ring $Z / nZ; Z / nZ$;

The evaluation function, which associates with a polynomial P with real coefficients value $P(c)$ a fixed real c . The rings consist morphisms between them, making the class of rings a category.

Some elementary properties of an R –module M :

- (i) $0m = 0, m \in M$
- (ii) $a0 = 0, a \in R$
- (iii) $(-a)m = -(am) = a(-m), a \in R, m \in M$,

Where 0 on the right side of (i) and (ii) is the zero of M , 0 on the left side of (i) is the zero of R .

To prove (i), consider $am = (a + 0)m = am + 0m$.

To prove (ii), consider $am = a(m + 0) = am + a0$.

To prove (iii) consider $0 = 0m = (a + (-a))m = am + (-a)m$, and also consider $0 = a0 = a(m + (-m)) = am + a(-m)$.

Throughout, all modules are left module unless otherwise stated.

Examples of modules

i. Let A be any additive abelian group. Then A is a left (also right) Z –module, because

$$k(a_1 + a_2) = ka_1 + ka_2, \quad (k_1 + k_2)a = k_1a + k_2a, (k_1k_2)a = k_1(k_2a),$$

$$1a = a,$$

For all integer $k, k_1, k_2 \in Z$ and for all $a, a_1, a_2 \in A$

Let R bearing. then R itself can be regarded as a left R –module by defining $am, m \in R, a \in R$. to be the product of a and m as elements of the ring R . Then the distributive laws and the associative law for multiplication in the ring R show that R is a left R –module .

Similarly, R is also a right R –module .

ii. Let M be the set of $m \times n$ matrices over a ring R .

Then M is a module over R , because M is an additive abelian group under the usual addition of matrices, and the usual scalar multiplication (ra_{ij}) of the matrices $(a_{ij}) \in M (a_{ij}) \in M$ by the element $r \in R$ satisfies axioms (i) – (iv) for a module.

In particular, the set of $1 \times n$ (or $n \times 1$) matrices –the set of n –tuples, denoted by R^n – is a module over R .

Further, by choosing $R = R$ and $n = 2$ (or 3), we obtain that the set of vector in a plane (or in space) forms a vector space over the field R .

iii. Let M and N be R –modules . In the Cartesian product $M \times N$, define

$$(x, y) + (x', y') = (x + x', y + y'),$$

$$r(x, y) = (rx, ry)$$

For all $x, x' \in M, y, y' \in N$, and $r \in R$. Then $M \times N$ becomes an R -module, called the direct product of the R -modules M and N .

iv. the polynomial ring $R[x]$ over R is an R -module.

v. Let R be a ring and let S denote the set of all sequences $(a_i), i \in \mathbb{N}, a_i \in R$. Define

$$(a_i) + (b_i) = (a_i + b_i)$$

$$(a_i) = (\alpha b_i),$$

Where $a, a_i, b_i \in R$. Then S is a left R -module.

SUB MODULES AND DIRECT SUMS

Definition

A nonempty subset N of an R -module, M is called an R -sub module (or simply submodule) of M if

- (i) $a - b \in N$ for all $a, b \in N$.
- (ii) $ra \in N$ for all $a \in N, r \in R$

Clearly; (0) or simply 0 and M are R -sub modules, called trivial sub modules. In case R is a field, N is a R -sub module of M , then N is also a R -module in its own right.

Examples of Sub modules

i. Each left ideal of a ring R is a R -module of the left R -module R , and conversely.

This follows from the definition of a left ideal.

ii. The subset $W = \{(\alpha, 0, 0) \mid \alpha \in F\}$ of F^3 is a subspace of the vector space F^3 .

iii. If M is an R -module and $x \in M$, then the set

$Rx = \{rx \mid r \in R\}$ is an R -submodule of M , for

$$r_1x - r_2x = (r_1 - r_2)x \in Rx,$$

$$r_1(r_2x) = (r_1r_2)x \in Rx, \text{ for all } r_1, r_2 \in R.$$

iv. If M is an R -module and $x \in M$, then the set

$$K = \{rx + nx \mid r \in R, n \in \mathbb{Z}\}$$

is an R -module of M containing x . further, if R has unity, then

$$K = Rx. \text{ First, } (K, +) \text{ is clearly an abelian subgroup of } (M, +).$$

Next, let $a \in R, rx + nx \in K$. Then

$$\begin{aligned} a(rx + nx) &= a(rx) + a(nx) \\ &= (an)x + a(x + \dots + x) \text{ or } arx + a((-x) + \dots + (-x)), \end{aligned}$$

According to whether n is a positive or a negative integer. But then

$$a(rx+nx) = ((ar)+a+\dots+a)x \text{ or } ((ar)+(-a)0+\dots+(-a))x,$$

Since $a(-x) = (-a)x$. Therefore ,

$$a(rx+nx) = ux \text{ for some } u \in R$$

Hence , $a(rx+nx) \in K$, for all a, r in R and for all n (including 0) in Z . choosing

$r = 0 \in R$ and $n = 1 \in Z$ in $rx+nx$ gives $x \in K$. It is worth noting that if L is any other R —*module* of M containing x , then L contains all elements of the form

$$rx+nx, r \in R, n \in Z ; \text{ hence, } K \subset L..$$

Thus, K is the smallest R —*sub module* of M containing x , usually denoted (x) . Suppose R has unity e .

then, $rx+nx \in Rx$. Similarly, if $n \leq 0$, then

$$rx+nx \in Rx. \text{ So } K \subset Rx. \text{ Trivially, } Rx \subset K.$$

Theorem

Let $(N_i)_{i \in \Lambda}$ be a family of R -submodules of an R -module M . Then $\bigcap_{i \in \Lambda} N_i$ is also an R -submodule.

Proof

Let $x, y \in \bigcap_{i \in \Lambda} N_i, a \in R$. Then for all $i \in \Lambda, x-y \in N_i$ and $ax \in N_i$, because N_i are R -submodules.

Then, $x-y, xy \in \bigcap_{i \in \Lambda} N_i$, which proves that $\bigcap_{i \in \Lambda} N_i$ is an R -module.

Let S be a subset of an R -module M . let $\mathcal{A} = \{N \mid N \text{ is an } R\text{-module of } M \text{ containing } S\}$. then $\mathcal{A} \neq \emptyset$ because $M \in \mathcal{A}$. let $K = \bigcap_{N \in \mathcal{A}} N$. Then K is the smallest R -module of M containing s and is denoted by (S) .

The smallest R -module of M containing a subset S is called the R -sub module generated by S . If $S = \{x_1, \dots, x_n\}$ is a finite set, then (S) is also written as (x_1, \dots, x_m) .

Definition

An R -module M is called a finite generated module if $M = (x_1, \dots, x_k)$ for some $x_i \in M, 1 \leq i \leq k$. The elements x_1, \dots, x_k are said to be generate M .

Example

The cyclic module generated by x is precisely $\{rx + nx \mid r \in R, n \in Z\}$, and R has unity then it implies to $\{rx \mid r \in R$

Theorem

If an R -module M is generated by a set $\{x_1, x_2, \dots, x_n\}$ and $1 \in R$, then

$M = \{r_1x_1 + r_2x_2 + \dots + r_nx_n \mid r_i \in R\}$. The right side is symbolically written as

$$\sum_{i=1}^n Rx_i$$

Proof

Clearly , if $m, m_1, m_2 \in \sum_{i=1}^n Rx_i$ and $r \in R$, then :

$$m_1 - m_2 \in \sum_{i=1}^n Rx_i .$$

Thus , all $x_i \in \sum_{i=1}^n Rx_i$

But since M is the smallest sub module of M containing all x_i , the sub module

$\sum_{i=1}^n R x_i$ must be equal to M .

If an element $m \in M$ can be expressed as $m = a_1 x_1 + \dots + a_n x_n, a_i \in R$

$x_i \in M, i = 1, \dots, n$ then we say that m is a linear combination of the elements

$x_1, \dots, x_n \in R$.

we remark that the set of generators of a module need not be unique.

For example, let S be the set of all polynomials in x over a field F of degree $\leq n$.

Then S is a vector space over F with $\{1, x, x^2, x^3, \dots, x^n\}$ and

$\{1, 1+x, x^2, x^3, \dots, x^n\}$ as two distinct sets of generators.

7 Definition

Let $(N_i), 1 \leq i \leq k$, be a family of R -submodules of a module M . then the sub module generated by $\cup_{i=1}^k N_i$

that is, the smallest sub-module containing the submodules $N_i, 1 \leq i \leq k$, is called the sum of sub modules $N_i, 1 \leq i \leq k$, and is

denoted by $\sum_{i=1}^k N_i$

Theorem

If $(N_i), 1 \leq i \leq k$, is a family of R -modules of a module M , then

$\sum_{i=1}^k N_i = \{x_1 + \dots + x_k \mid x_i \in N_i\}$.

Proof

Let $S = \{x_1 + \dots + x_k \mid x_i \in N_i\}$. If $x_1 + \dots + x_k$ and $y_1 + \dots + y_k$ belong to S then

$$(x_1 + \dots + x_k) - (y_1 + \dots + y_k) = (x_1 - y_1) + \dots + (x_k - y_k)$$

also belong to S , because $x_i - y_i \in N_i, 1 \leq i \leq k$. Also, if $r \in R$, then

$$r(x_1 + \dots + x_k) = r x_1 + \dots + r x_k \in S,$$

because each $r x_i, 1 \leq i \leq k$, is in N_i . Thus, S is a left R -submodule.

Further, if K is any left R -sub module that contains each sub module N_i , then K contains all elements of the form $x_1 + \dots + x_k, x_i \in N_i$. Thus, K contains S .

Hence, S is the smallest sub module containing each $N_i, 1 \leq i \leq k$.

Therefore, by definition of $\sum_{i=1}^k N_i$

Problem 2.

Prove that if M is an R -module, then $Z(M)$ is a submodule of M and $Z(R)$ is a proper two-sided ideal of R . In particular, if R is a simple ring, then $Z(R) = \{0\}$.

Solution. First note that $0 \in Z(M)$ because $\text{ann}(0) = R \subseteq_e R$. Now suppose that $x_1, x_2 \in Z(M)$. Then $\text{ann}(x_1 + x_2) \supseteq \text{ann}(x_1) \cap \text{ann}(x_2) \subseteq_e M$ by Problem 1. Therefore $\text{ann}(x_1 + x_2) \subseteq_e M$ and hence $(x_1 + x_2) \in Z(M)$. Now let $r \in R$ and $x \in Z(M)$. We need to show that $rx \in Z(M)$. Let J be a nonzero left ideal of R . Then Jr is also a left ideal of R . If $Jr = \{0\}$ then $J \subseteq \text{ann}(rx)$ and thus $\text{ann}(rx) \cap J = J \neq \{0\}$. If $Jr \neq \{0\}$. Then $\text{ann}(rx) \cap Jr \neq \{0\}$ because $x \in Z(M)$. So there exists $s \in J$ such that $sr \neq 0$ and $srx = 0$. Hence $0 \neq s \in \text{ann}(rx) \cap J$. So $rx \in Z(M)$ and thus $Z(M)$ is a submodule of M . Now, considering R as a left R -module, $Z(R)$ is a left ideal of R , by what we have just proved. To see why $Z(M)$ is a right ideal, let $r \in R$ and $x \in Z(R)$. Then $\text{ann}(xr) \supseteq \text{ann}(x) \subseteq_e R$ and so $\text{ann}(x) \subseteq_e R$ i.e. $xr \in Z(R)$. Finally, $Z(R)$ is proper because $\text{ann}(1) = \{0\}$ and so $1 \notin Z(R)$.

Problem 3. Prove that if $M_i, i \in 1$ are R -modules, then $Z(\bigoplus_{i \in 1} M_i) = \bigoplus_{i \in 1} Z(M_i)$. Conclude that if R is a semisimple ring, then $Z(R) = \{0\}$.

Solution. The first part is a trivial result of Problem 1 and this fact that if $x = x_1 + x_2 + \dots + x_n$ where the sum is direct, then $\text{ann}(x) = \bigcap_{i=1}^n \text{ann}(x_i)$. The second now follows trivially from the first part, Problem 2 and the Wedderburn-Artin theorem.

Problem 4. Suppose that R is commutative and let $N(R)$ be the nilradical of R . Prove that

- 1) $N(R) \subseteq Z(R)$;
- 2) it is possible to have $N(R) \not\subseteq Z(R)$
- 3) if $Z(R) \neq \{0\}$ then $N(R) \subseteq_e Z(R)$ as R -modules or $Z(R)$ -modules.

Solution. 1) Let $a \in N(R)$. Then $a^n = 0$ for some integer $n \geq 1$. Now suppose that $0 \neq r \in R$. Then $ra^n = 0$. Let $m \geq 1$ be the smallest integer such that $ra^m = 0$. Then $0 \neq ra^{m-1} \in \text{ann}(a) \cap Rr$ and hence $a \in Z(R)$.

2) Let $R_i = \mathbb{Z}/2^i\mathbb{Z}, i \geq 1$ and put $R = \prod_{i=1}^{\infty} R_i$. For every i , let $a_i = 2 + 2^i\mathbb{Z}$ and consider $a = (a_1, a_2, \dots) \in R$. It is easy to see that $a \in N(R) \setminus Z(R)$.

3) Let $a \in Z(R) \setminus N(R)$. Then $\text{ann}(a) \cap Ra \neq \{0\}$ and thus there exists $r \in R$ such that $ra \neq 0$ and $ra^2 = 0$. Hence and so $ra \in N(R)$. Thus $0 \neq ra \in N(R) \cap Ra$ implying that $N(R)$ is an essential R -submodule of $Z(R)$. Now, we view $Z(R)$ as a ring and we want to prove that $N(R)$ as an essential ideal of $Z(R)$. Again, let $a \in Z(R) \setminus N(R)$. Then $\text{ann}(a) \cap Ra^2 \neq \{0\}$ and thus there exists $r \in R$ such that $ra^2 \neq 0$ and $ra^3 = 0$. Let $s = ra \in Z(R)$. Then $(sa)^2 = 0$ and thus $0 \neq sa \in N(R) \cap Z(R)$ implying that $N(R)$ is an essential ideal of $Z(R)$.

Problem 1. (Richard Brauer) Let R be a negligible left perfect of a ring R . At that point either $I = Re$ or for some non-zero idempotent $e \in I$

Arrangement. Assume that $I^2 \neq \{0\}$. At that point there exists some $x \in I$ such that $Ix \neq \{0\}$. In this manner $Ix = I$, in light of the fact that $Ix \subseteq I$ is a non-zero remaining perfect of R and is an insignificant left perfect of R . Hence there exists $0 \neq e \in I$ such that $ex = x$ thus

$$(e^2 - e)x = 0 \quad (1)$$

Then again, $J = \{r \in I; rx = 0\}$ is clearly a left perfect of R which is contained in I . Since $Ix \neq \{0\}$ we have $J \neq I$ and in this manner, by the insignificance of I , we should have $J = 0$. In this manner $e^2 - e = 0$, by (1). So $e \in I$ is a non-zero idempotent. Presently Re is a left perfect of R which is contained in I . Also $0 \neq e = e^2 \in Re$ thus $Re \neq \{0\}$. Therefore $Re = I$, by the negligibility of I .

Take note of that we didn't require R to have 1. Likewise, a comparable outcome holds for negligible right standards of R , i.e. in the event that I is a negligible right perfect of R , then either $I^2 = \{0\}$ or $I = eR$ for some non-zero idempotent $e \in I$. In the event that R is a semisimple ring (with 1), for instance full lattice rings over division algebras, then every left (resp. right) perfect of R is created by some idempotent, as the following issue appears.

Problem 2. Give R a chance to be a semisimple ring and I any left (resp. right) perfect of R . At that point there exists some idempotent $e \in I$. to such an extent that $I = Re$ (resp. $I = eR$).

Solution: We'll just demonstrate the claim for left beliefs of R . The evidence for right standards is comparable. Since R is semisimple, there exists a left perfect J of R with the end goal that $R = I \oplus J$. So $1 = e + f$ for some $e \in I$ and $f \in J$. Thus $e = e^2 + ef$ thus $e^2 = e$ and $ef = 0$, in light of the fact that $e, e^2 \in I$ and $ef \in J$ and the entirety is immediate. So e is an idempotent. It is clear that $Re \subseteq I$. Presently if $x \in I$, then $x = xe + xf$ and along these lines $x = xe$. In this way $I \subseteq Re$ and we're finished.

JACOBSON SEMI-SIMPLE OR PRIMITIVE SEMI-SIMPLER RING

A ring is called semi primitive or Jacobson semi basic if its Jacobson radical is the zero perfect. A ring is semi primitive if and just on the off chance that it has an unwavering semi straightforward left module. The semi primitive property is left-right symmetric, thus a ring is semi primitive if and just in the event that it has a reliable semi straightforward right module. A ring is semi primitive if and just in the event that it is a sub coordinate result of left primitive rings. A commutative ring is semi primitive if and just on the off chance that it is a sub coordinate result of fields, A left artinian ring is semi primitive if and just in the event that it is semi straightforward, Such rings are here and there called semi basic Artinian

Examples

The ring of whole numbers is semi primitive, yet not semi basic. Each primitive ring is semi primitive. The result of two fields is semi primitive however not primitive. Each von Neumann customary ring is semi primitive.

Theorem. Let M be a semisimple R -module and $N \subseteq M$ a submodule. Then we can find simple submodules $M_i \subseteq M$ (indexed by $i \in I$) such that

$$M = N \oplus (\oplus_{i \in I} M_i)$$

The “direct sum” \oplus means that every element m of M is uniquely writable as a sum $n + \sum_i m_i$ where $n \in N, m_i \in M_i$ and only finitely many terms are non-zero.

Proof

This will be by Zorn’s lemma. Consider collections Σ of simple submodules S of M such that:

$$M_\Sigma := N \oplus (\oplus_{s \in \Sigma} S)$$

is a direct sum. Note that at least one Σ exists, i.e. $\Sigma = \emptyset$ is valid (in which case we get $M_\emptyset = N$). [For those who worry about set-theoretic validity, note that the collection of all such Σ forms a bona fide set.]

To apply Zorn’s, we need to prove that every chain of Σ ’s has an upper bound.

Suppose $\{\sum_\alpha\}_\alpha$ is a chain: i.e. for any \sum_α, \sum_β , either $\sum_\alpha \subseteq \sum_\beta$ or $\sum_\beta \subseteq \sum_\alpha$. Let $\Sigma = U_\alpha \sum_\alpha$ let us show that $M_\Sigma = N \oplus (\oplus_{s \in \Sigma} S)$ is a direct sum.

- If not, then $n + \sum_{s \in \Sigma} m_s = 0$ for some $n \in N, m_s \in S$. But this is a finite sum, so the equality already holds in some \sum_α (since the \sum_α ‘s form a chain), which is a contradiction.

Thus, the chain $\{\sum_\alpha\}_\alpha$ has an upper bound. Zorn’s lemma tells us there is a maximal Σ . If $M_\Sigma \neq M$, pick $m \in M - M_\Sigma$. Since M is a sum of simple submodules, write

$$m = m_1 + m_2 + \dots + m_r, m_k \in M_k \text{ where each } M_k \text{ is simple.}$$

Since $m \in M_\Sigma$ we have $M_k \not\subseteq M_\Sigma$ for some k . But this means $M_k \cap M_j$ is a proper submodule of M_k , and must be zero (since M_k is simple). Hence $M_k \oplus M_\Sigma$ is a direct sum, so we could have added the simple module M_k to the collection Σ , contradicting its maximality. Thus, $M_\Sigma = M$ and we're done. ♦

Now we're ready to prove all the necessary properties of semisimple modules.

Corollary 1. Every semisimple module M is a direct sum of simple modules.

Proof. Pick $N = 0$ in the theorem. ♦

Corollary 2. If each $N_i \subseteq M$ is a semisimple submodule of a module M , then so is $N := \sum N_i$

Proof. Each N_i is a sum of simple modules; by definition so is N . ♦

Corollary 3. If N is a submodule of a semisimple M , then there is a submodule P of M such that $M = N \oplus P$.

Proof. Apply the theorem and let $P := \bigoplus_{i \in I} M_i$. ♦

Corollary 4. A submodule and quotient of a semisimple module M is semisimple.

Proof. Submodule follows from the definition of semisimplicity; quotient follows from the theorem, since $\frac{M}{N} \cong \bigoplus_{i \in I} \frac{M_i}{N_i}$ is a direct sum of simple modules. ♦

Theorem. Any module over a semisimple ring R is semisimple.

Proof

Let M be a module. If m is a non-zero element of M , take the homomorphism $f: R \rightarrow M$, which takes $r \rightarrow rm$. Then Rm is a submodule of M isomorphic to $R/\ker(f)$, which is a semisimple R -module since R is. Thus Rm is semisimple. Since M is a sum of semisimple submodules, M is also semisimple. ♦

Let us look at some ways to create semisimple rings.

Proposition. (i) If I is a (two-sided) ideal of semisimple ring R , then R/I is a semisimple ring.

(ii) If R and S are semisimple rings, so is $R \times S$.

Pr(i) Any left ideal of R/I corresponds to a left ideal of R containing I , which is a sum of simple submodules J . The image of J in R/I , i.e. $(J + I)/I \cong J / (J \cap I)$, is thus either 0 or J . Either way, R/I is a sum of simple submodules.

(ii) Any left ideal M of $R \times S$ is of the form $I \times J$, for left ideal I of R and J of S . [To see why, multiply elements of M by $(1, 0)$ and $(0, 1)$.] Since I and J are both sums of simple submodules, so is $I \times J$. ♦

Finally, decomposing R gives us a complete list of simple R -modules.

Proposition. Give R a chance to be a semisimple ring; compose as an immediate total of straightforward left standards. At that point any straightforward module M is isomorphic to some . Specifically, there are just limitedly numerous straightforward R -modules up to isomorphism.

Proof

We know that any simple module M is isomorphic to quotient R/I for a maximal left ideal I of R . For each i , consider

$$f_i : N_i \rightarrow \oplus_i N_i \rightarrow \frac{(\oplus_i N_i)}{I} = M$$

Since N_i, M are both simple, $f_i = 0$ or an isomorphism. If all $f_i = 0$, then so is $\sum_i f_i : R = \oplus_i N_i \rightarrow M$ which is absurd. Hence some f_i is an isomorphism, which proves the first statement.

The second statement follows from the following lemma. ♦

Lemma. Writing the base ring as a direct sum of submodules $R = \oplus_i N_i$ only finitely many of the modules are non-zero.

Proof

Indeed, write 1 as a finite sum $x_1 + x_2 + \dots + x_k$ where $x_i \in N_i$ For an N_i not in this list, any $y \in N_i$ gives:

$$y = y.1 = yx_1 + yx_2 + \dots + yx_k \in N_1 + N_2 + \dots + N_k$$

So $y = 0$ since N_i does not lie in the list of N_1, \dots, N_k

THEOREM (SIMPLE STATEMENT)

let M be a simple R -module . then $HomR(M, M)$ is a divisionring.

Proof

Let N be the kernel of ϕ . The subspace E is not entirely because ϕ is not zero. It is stable by any application u of U

$$\forall u \in U \exists v \in L(F), \phi \circ u(N) = v \circ \phi(N) = v \circ \phi(N)$$

Therefore $u(N)$ is included in N . For Toughness U, N is reduced to $\{0\}$.

Let M the image of ϕ . The subspace F is not reduced to $\{0\}$ because ϕ is not zero. It is stable by any

application v of V

$$\forall v \in V \exists u \in L(F), v \circ \phi(E) = \phi \circ u(E) \subset \phi(E)$$

Therefore $v(M)$ is included in M . For irreducibility of V , M is equal to F .

4.3 COLLARIES:

Corollary 1

Let E be a vector space of finite dimension over an algebraically closed field K and U an irreducible part of $L(E)$. If an endomorphism ϕ of E commutes with every element of U , then ϕ is a scaling.

Proof

If Id is the application identity, we have:

$$\forall \lambda \in K, \forall u \in U (\phi - \lambda Id) \circ u = u \circ (\phi - \lambda Id)$$

It is deduced by applying the Schur lemma that $\phi - \lambda Id$ is an auto morphism or is zero. * λ is an eigenvalue of ϕ , then $\phi - \lambda * Id$ is not an auto morphism, so is the null application, which proves the corollary.

In the case of the representation of a group of finite exponent e , then any auto morphism of the image has to annihilator polynomial $X^e - 1$. Therefore, if the polynomial is split over K , the corollary still applies.

Corollary 2

Any irreducible representation of an Abelian group in a finite-dimensional space over an algebraically closed field is 1 degree.

For let (E, ρ) such representation and D a straight E . whatever the element s of the group, ρs commutes with all endomorphism of representation. By Corollary 1, s is a dilation. Thus, D is therefore invariant equal to E .

For finite groups

Corollary 3

Let (E, ρ_E) and (F, ρ_F) two irreducible representations of G over a field K whose characteristic does not divide the order of the group g and where the polynomial $X^g - 1$ split, and a linear mapping of ψ E to F , we define the linear map ϕ from E to F by:

$$\phi = \frac{1}{g} \sum_{s \in G} \rho_F(s) \circ \psi \circ \rho_E(s)^{-1}$$

If the representations are not isomorphic, then ϕ is zero.

If representations are equal, then ϕ is a scaling ratio $(1/n) T_r(\psi)$.

Proof

Check at first that ϕ satisfies the following property:

$$\forall t \in G \quad \phi \circ \rho_E(t) = \rho_F(t) \circ \phi \quad \text{ou encore} \quad \phi = \rho_F(t) \circ \phi \circ \rho_E(t)^{-1}$$

Note first that if t is an element of G , the application of G in G which associates to t s is a permutation of G . We deduce that:

$$\forall t \in G \rho_F(t) \circ \varphi \circ \rho_E(t)^{-1} = \frac{1}{g} \sum_{s \in G} \rho_F(t) \circ \rho_F(s)^{-1} \circ \rho_E(t)^{-1} = \frac{1}{g} \sum_{s \in G} \rho_F(ts) \circ \psi \circ \rho_E(ts)^{-1} = \varphi$$

1. As the performances are not isomorphic, φ cannot be both one and onto. Schur's lemma shows that, as φ is not an auto morphism φ is the nullapplication.
2. If $(E, \rho_E) = (F, \rho_F)$, the assumptions of Corollary 1 hold, which shows that φ is a scaling. In this case, the expression defining φ is the average of all similar applications ψ g and thus having the same record as ψ . The traces of ψ and φ are equal. Noting λ the ratio of homothetic φ we have: $n\lambda = \text{Tr}(\varphi) = \text{Tr}(\psi)$. Applying all this to an arbitrary ψ trace 1, there are more than n is invertible in K .

Note.

If the characteristic of K p is nonzero, the proof of this corollary shows that the prime p does not divide n . As it was assumed that p does not divide g , this is not surprising when you consider that the degree n of an irreducible representation g always divides the order of the group.

Corollary 4

It is a fourth corollary that is used in the character theory. It is the translation in terms of matrices of the previous corollary. Use the following notations:

A and B are two matrix representations of a finite group G of order g on the same field K whose characteristic does not divide g and where the polynomial

$X^g - 1$ is cleaved. The respective dimensions of E and F are shown as n and m . The image of an element $s \in G$ to A (resp. B) is denoted $a_{ij}(s)$ (resp. $b_{ij}(s)$).

We have then under the assumptions of Corollary:

1. If the representations A and B are not isomorphic, then:

$$\forall i, j \in [1, n] \quad \forall k, l \in [1, m] \quad \sum_{s \in G} a_{ij}(s) b_{kl}(s^{-1}) = 0$$

2. Noting δ_{ij} the Kronecker symbol, then:

$$\forall i, j, k, l \in [1, n] \quad \frac{1}{g} \sum_{s \in G} a_{ij}(s) b_{kl}(s^{-1}) = \frac{1}{n} \delta_{il} \delta_{jk}$$

Proof

1. If C a matrix size $m \times n$ coefficients (c_{jk}) , translation of point 1 of the previous corollary show that:

$$\sum_{s \in G} A(s) C B(s)^{-1} = 0$$

Therefore

$$\forall i \in [1, n], \forall l \in [1, m] \sum_{jk} \sum_{s \in G} a_{ij}(s) c_{jk} b_{kl}(s)^{-1} = \sum_{jk} \left(\sum_{s \in G} a_{ij}(s) b_{kl}(s)^{-1} \right) c_{jk} = 0$$

This equality is true for any matrix C, so for any value of C_{jk} , demonstrating the point 1.

2. With the same notation (now A and B = m = n), we get from point 2 of the previous corollary:

$$\frac{1}{g} \sum_{s \in G} A(s) C A(s)^{-1} = \frac{1}{n} \text{Tr}(C) Id$$

Therefore

$$\forall i, j \in [1, n] \frac{1}{g} \sum_{jk} \sum_{s \in G} a_{ij}(s) c_{jk} b_{kl}(s)^{-1} = \frac{1}{n} \sum_k c_{kk} \delta_{il}$$

We can deduce:

$$\forall i, j, k, l \in [1, n] \frac{1}{g} \sum_{jk} a_{ij}(s) a_{kl}(s)^{-1} = \frac{1}{n} \delta_{il} \delta_{jk},$$

And point 2 is proved.

CONCLUSIONS

The review finishes up the range of abstract algebra based math known as module hypothesis also known as module theory, a semisimple module or totally reducible module is a kind of module that can be seen effectively from its parts. A ring that is a semisimple module over itself is known as an Artinian semisimple ring. Some vital rings, for example, matrix rings of limited dimensions over fields of trademark zero, are semisimple rings. An Artinian ring is at first comprehended by means of its biggest semisimple remainder. The structure of Artinian semi-simple rings is surely knew by the Artin–Wedderburn hypothesis, which displays these rings as limited direct results of framework rings. The straightforward modules over a ring R are the (left or right) modules over R that have no non-zero appropriate submodules. Comparably, a module M is straightforward if and just if each cyclic submodule produced by a non-zero component of

M equivalents M. Straightforward modules shape building hinders for the modules of limited length, and they are closely resembling the basic gatherings in group theory.

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