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A Study on Integral Equation and Its Applications

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ABSTRACT:

The present survey paper samples recent advances in the numerical analysis of Volterra integral equations of the first and second kind and of integro-differential equations (including equations with weakly singular kernels); except for some important earlier references the discussion focuses on the development which has taken place during the last dozen years. A fairly extensive bibliography (selected to be representative rather than comprehensive) complements the paper.

The study also presents the Integral transform, mathematical operator that produces a new function f(y) by integrating the product of an existing function F(x) and a so-called kernel function K(x, y) between suitable limits. The process, which is called transformation, is symbolized by the equation f(y) = ∫K(x, y)F(x)dx. Several transforms are commonly named for the mathematicians who introduced them: in the Laplace transform, the kernel is e−xy and the limits of integration are zero and plus infinity; in the Fourier transform, the kernel is (2π)−1/2e−ixy and the limits are minus and plus infinity. The study analysis the volterra integral equation and kernel types and presents the examples solution for the discussed problems.

Keywords: Volterra integral equations

1 INTEGRAL EQUATION:

Any functional equation in which the unknown function appears under the sign of integration is called an integral equation. Integral equations arise in a great many branches of science; for example, in potential theory, acoustics, elasticity, fluid mechanics, radiative transfer, theory of population, etc. In many instances the integral equation originates from the conversion of a boundary-value problem or an initial-value problem associated with a partial or an ordinary differential equation, but many problems lead directly to integral equations and cannot be formulated in terms of differential equations. Integral equations are of many types; here we attempt to indicate some of the main distinguishing features with particular regard to the use and construction of algorithms.

Integral transform, mathematical operator that produces a new function $f(y)$ by integrating the product of an existing function $F(x)$ and a so-called kernel function $K(x, y)$ between suitable limits. The process, which is called transformation, is symbolized by the equation $f(y) = \int K(x, y)F(x)dx$. Several transforms are commonly named for the mathematicians who introduced them: in the Laplace transform, the kernel is e^{-xy} and the limits of integration are zero and plus infinity; in the Fourier transform, the kernel is $(2\pi)^{-1/2}e^{-ixy}$ and the limits are minus and plus infinity.

Integral transforms are valuable for the simplification that they bring about, most often in dealing with differential equations subject to particular boundary conditions. Proper choice of the class of transformation usually makes it possible to convert not only the derivatives in an intractable differential equation but also the boundary values into terms of an algebraic equation that can be easily solved. The solution obtained is, of course, the transform of the solution of the original differential equation, and it is necessary to invert this transform to complete the operation. For the common transformations, tables are available that list many functions and their transforms.

The precursor of the transforms were the Fourier series to express functions in finite intervals. Later the Fourier transform was developed to remove the requirement of finite intervals.

Using the Fourier series, just about any practical function of time (the voltage across the terminals of an electronic device for example) can be represented as a sum of sines and cosines, each suitably scaled (multiplied by a constant factor), shifted (advanced or retarded in time) and "squeezed" or "stretched" (increasing or decreasing the frequency). The sines and cosines in the Fourier series are an example of an orthonormal basis (Doetsch, G. (2012)).

1.1.1 DERIVATION:

If you have a function ***f*(x)** and a function **k(x,s)** then you can (as long as the product of ***f*(x)** times **k(x,s)** is integrable on the set **X**) always form another function of a new variable s as follows:

$$F(s) = \int_X k(x,s) f(x) dx$$

We have just "transformed" the function f(x) into the function F(s) via an "integral transform." Why the hell would anyone want to do this? Well, the function ***F(s)*** is sometimes easier to work with than ***f(x)*** itself, or tells us interesting information about ***f(x)*** that it would be hard to figure out in other ways.

Of course, the interpretation of this new function ***F(s)*** will depend on what the function ***k(x,s)*** is. Choosing ***k(x,s) = 0***, for example, will mean that ***F(s)*** will always be zero. This is pretty boring and tells us nothing about ***f(x).***

Whereas choosing $k(x,s) = x^s$ will give us the sth moment of ***f(x)*** whenever ***f(x)*** is a probability density function. For s=1 this is just the mean of the distribution ***f(x)***. Moments can be really handy.

A particularly interesting class of functions k(x,s) are ones that produce invertible transformations (which implies that the transform destroys no information contained in the original function) (Bracewell,

R. (1965)). This will occur when there exists a function **K(x,s)** (the inverse of **k(x,s))** and a set S such that

$$f(x) = \int_S K(x,s)F(s)ds$$

That undoes the original transformation (or, at least, undoes it for some large class of functions **f(x)).**

Whenever this is the case, we can view our operation as changing the domain from x space to s space. Each function **f** of **x** becomes a function **F** of **s** that we can convert back to **f** later if we so choose to. Hence, we're getting a new way of looking at our original function!

It turns out that the Fourier transform, which is one of the most useful and magical of all integral transforms, is invertible for a large class of functions. We can construct this transformation by setting:

$$k(x,s) = e^{-ixs}$$
$$K(x,s) = e^{ixs}$$

which leads to a very nice interpretation for the variable s. We call **F(s)** in this case the "Fourier transform of **f**", and we call s the "frequency". Why is s frequency? Well, we have Euler's famous formula:

$$e^{ixs} = \cos(xs) + i\sin(xs)$$

So modifying s modifies the oscillatory frequency of **cos(xs)** and **sin(xs)** and therefore of **k(x,s).** There is another reason to call s frequency though. If x is time, then **f(x)** can be thought of as a waveform in time, and in this case **|F(s)|** happens to represent the strength of the frequency s in the original signal. You know those bars that bounce up and down on stereo systems? They take the waveforms of your music, which we call **f(x),** then apply (a discrete version of) the Fourier transform to produce **F(s).** They then display for you (what amounts to) the strength of these frequencies in the original sound, which is **|F(s)|.** This is essentially like telling you how strong different notes are in the music sound wave (Tranter, C. J. (1951)).

Below are a few other neat examples of integral transform.

The Laplace transform:

$$k(x,s) = e^{-xs}$$

This is handy for making certain differential equations easy to solve (just apply this transformation to both sides of your equation!)

The Hilbert transform:

$$k(x, s) = \frac{1}{\pi}\frac{1}{x - s}$$

This has the property that (under certain conditions) it transforms a harmonic function into its harmonic conjugate, elucidating the relationship between harmonic functions and holomorphic functions, and therefore connecting problems in the plane with problems in complex analysis.

The identity transforms:

$$k(x, s) = \delta(x - s)$$

Here $\delta$is the dirac delta function. This is the transformation that leaves a function unchanged, and yet it manages to be damn useful.

1CHECKING FOR SOLUTIONS

The kernel trick avoids the explicit mapping that is needed to get linear learning algorithms to learn a nonlinear function or decision boundary. For all X and X' in theinput space χ, certain functions $k(X, X')$ can be expressed as an inner product inanother space v. The function $k : \chi \times \chi \to R$ is often referred to as a kernel or akernel function. The word "kernel" is used in mathematics to denote a weighting function for a weighted sum or integral.

Certain problems in machine learning have additional structure than an arbitrary weighting function k. The computation is made much simpler if the kernel can be written in the form of a "feature map" $\varphi : \chi \to v$ whichsatisfies

$$k(x, x') = \langle \varphi(x), \varphi(x') \rangle v$$

The key restriction is that $\langle .,. \rangle v$ must be a proper inner product. On the other hand, an explicit representation for φ is not necessary, as long as v is an inner product space. The alternative follows from Mercer's theorem: an implicitly defined function φ exists whenever the space χ can be equipped with a suitable measure ensuring the function k satisfies Mercer's condition.

Mercer's theorem is akin to a generalization of the result from linear algebra that associates an inner product to any positive-definite matrix. In fact, Mercer's condition can be reduced to this simpler case. If we choose as our measure the counting $\mu(T) = |T|$ for all $T \subset X$,which counts the number of points inside the set *T*, then the integral in Mercer's theorem reduces to a summation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} k(X_i, X_j) c_i c_j \geq 0.$$

If this summation holds for all finite sequences of points $\{X_1, ...X_n\}$ in χ and allchoices of n real-valued coefficients $(c_1, ..., c_n)$ (cf. positive definite kernel), then the function k satisfies Mercer's condition.

Some algorithms that depend on arbitrary relationships in the native space χ would, in fact, have a linear interpretation in a different setting: the range space of φ. The linear interpretation gives us insight about the algorithm. Furthermore, there is often no need to compute φ directly during computation, as is the case with support vector machines. Some cite this running time shortcut as the primary benefit. Researchers also use it to justify the meanings and properties of existing algorithms.

Theoretically, a Gram matrix $K \in R^{n\times n}$ with respect to $\{X_1, ...X_n\}$ (sometimesalso called a "kernel matrix"), where $K = (k(X_i, X_j))_{ij}$ positive semi-definite(PSD). Empirically, for machine learning heuristics, choices of a function k that do not satisfy Mercer's condition may still perform reasonably if k at least approximates the intuitive idea of similarity. Regardless of whether k is a Mercer kernel, k may still be referred to as a "kernel".

If the kernel function k is also a covariance function as used in Gaussian processes, then the Gram matrix K can also be called a covariance matrix.

Finally, suppose that $K$is a square matrix. Then $K^T K$ is a positive-semi-definite matrix.

KERNEL TYPES AND EXAMPLES

3.2.1. Symmetric Kernel: When the kernel $K(x,t)K(X,t)$ is symmetric or complex symmetric or Hermitian, if

$$K(x,t) = K^{-}(t,x)K(X,t) = K^{-}(t,x)$$

Where bar $K^{-}(t,X)$

Denotesthe complex conjugate of *K (t, x)*.That's if there is no imaginary part of the kernel then $K(x,t) = K(t,x)$ implies that*K* is a symmetric kernel.

For example

$K(x,t) = \sin(x+t)K(X,t)\sin(X+t)$ is symmetric kernel.

3.2.2 Separable or Degenerate Kernel: A kernel *K(x,t)* is called separable if it can be expressed as the sum of a finite number of terms, each of which is the product of 'a function' of *x* only and 'a function' of *t* only, i.e.,

$$K(X,t) =$$

$$K(x,t) = \sum_{n=1}^{\infty} i(x)\psi i(t)_t$$

3.2.3 Difference Kernel:

When $k(x,t) = k(x-t)$, the kernel is called difference kernel.

Resolvent or Reciprocal Kernel: The solution of the integral equation

$$y(x) = f(x) + \lambda \int_a^t k(x,t)y(t)dt$$

is of the form

$$y(x) = f(x) + \lambda \int_a^t k(x,t;\lambda) f(t)dt$$

kernel R$(x,t;\lambda)$ of the solution is called resolvent or reciprocal kernel.

3.3 Resolvent Kernal Of Volterra Integral Equation

Find solution of volterra integral equation by re solvent kernel by using integrate kernel?

we know that the volterra integral equation of 2nd kind is

$$\phi(x) = f(x) + \lambda\int_0^x k(x,t)\phi(t)dt \text{..........(1)}$$

Where $k(x,t)$ is a continuous function for $0 \le x \le a,\ 0 \le t \le x$ & $f(x)$ is continuous function for $0 \le x \le a$.
Now, we shall find the solution of integral equation (1) in the form of an infinite power series is λ as

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \dots\dots + \lambda^n\phi_n(x) + \dots.(2)$$

Substituting (2) in (1) we get

$$\phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \dots\dots + \lambda^n\phi_n(x) = \mathrm{f(x)} +$$
$$\lambda\int_0^2 k(x,t)\left[\phi_0(\mathrm{t}) + \lambda\phi_1(\mathrm{t}) + \lambda^2\phi_2(\mathrm{t}) + \dots\dots + \lambda^n\phi_n(\mathrm{t}) + \dots..\right] \text{......(3)}$$

Comparing the coefficients of like of 'λ'

$$\phi_0(x) = f(x)$$

$$\phi_1(x) = \int_0^x k(x,t)\phi_0(\mathrm{t})\,\mathrm{dt} = \int_0^x k(x,t) f(\mathrm{t})\,\mathrm{dt} \quad \text{.....(4)}$$

$$\because \phi_0(x) = f(x)$$

$$\phi_2(x) = \int_0^x k(x,t)\phi_1(\mathrm{t})\,\mathrm{dt}$$

$$\phi_2(x) = \int_0^x k(x,t)\left[\int_0^t k(t,t_1) f(t_1)dt_1\right]\mathrm{dt} \quad \text{........(5)}$$

$\vdots$

$\vdots$

The relation (4) & (5) yield a method how successive determination of the $\phi_n(x)$.

It may be show that the assumption made with respect to *f(x) & k(x)*. The series (2) converges uniformly in *x* and *y* for any λ and its sum is a unique solution of equation (1)

From (4) & (5)

$$\phi_1(x) = \int_0^x F(x,t) f(t)dt$$

$$\phi_2(x) = \int_0^x k(x,t)\left[\int_{t_1}^x k(t,t_1) f(t_1)dt_1\right]\mathrm{dt}$$

$$= \int_0^x f(t_1)dt_1\int_{t_1}^x k(x,t)k(t,t_1)\,\mathrm{dt}$$

$$= \int_0^x k_2(x,t_1) f(t_1)dt_1 \quad \text{........(6)}$$

Where $k_2(x,t) = \int_{t_1}^x k(x,t)k(t,t_1)dt \quad \text{......(7)}$

Similarly $\phi_n(x) = \int_0^x k_n(x,t) f(t)dt \quad (n = 1,2,3,\dots\dots) \quad (8)$

The functions $k_n(x,t)$ are called iterated kernels.

They are determined with the aid of recursion formula

$$k_1(x,t) = k(x,t)$$

$$k_{n+1}(x,t) = \int_t^x k(x,z) k_n(z,t) dz \qquad (n = 1,2,3,.....m) \qquad (9)$$

i.e.., $k_2(x,t) = \int_t^x k(x,z) k_1(z,t) dz = \int_t^x k(x,z) k(z,t) dz$

By using (8) & (9), the series (2) may be written as

$$\phi(x) = f(x) + \sum_{v=1}^{\infty} \lambda^v \int_0^x k_v(x,t) f(t) dt \qquad ....(10)$$

The function $R(x,t,\lambda) = \sum_{v=0}^{\infty} \lambda^v k_{v+1}(x,t)$ ...... (11)

(11) is called the Resolvent Kernel for the integral equation (1)
The series (11) converges absolutely & uniformly in the case of a continuous kernel *k(x, t)*
With the aid of the resolvent kernel the solution of the integral equation (1) is in the form

$$\phi(x) = f(x) + \lambda \int_0^x R(x,t,\lambda) f(t) dt \qquad (12)$$

$$\phi(x) = f(x) + \lambda \int_0^x k(x,t) \phi(t) dt$$

Example (1):-

Find the resolvent kernel of the volterra integral equation with kernel $k(x,t) = 1$

Solution:-

The iterated kernel are

$$k(x,t) = k(x,t) = 1.$$

$$\& \ \mathrm{k}_{n+1}(x,t) = \int_t^x k(x,\mathrm{z}) \mathrm{k}_n(z,t) dz \qquad (n = 1,2,...)$$

$$k_2(x,t) = \int_t^x k(x,z) k_1(z,t) dt \quad = \int_t^x 1.1.dz = [z]_t^x = x - t$$

$$k_3(x,t) = \int_t^x k(x,z) k_2(z,t) dz = \int_t^x 1.(z,t) dz$$

$$= \frac{z^2}{2} - tz]_t^x = \frac{x^2}{2} - tx - \frac{t^2}{2} + t^2$$

$$= \frac{x^2 - 2tx + t^2}{2} = \frac{(x-t)^2}{2}$$

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$$k_4(x,t)=\int_t^x k(x,z)k_3(z,t)dz=\int_t^x \frac{\left(z-t\right)^2}{2}dz$$

$$=\frac{1}{2}\left[\frac{z^3}{3}-2t\frac{z^2}{2}+t^2z\right]_t^x$$

$$=\frac{\left(x-t\right)^3}{3!}$$

Simplifying, we get

By definition of resolvent kernel

$$R(x,t,\lambda)=\sum_{n=0}^{\infty}\lambda^n k_{n+1}(x,t)\ \&\ \sum_{v=0}^{\infty}k_{v+1}(x,t)$$

$$=\sum_{n=0}^{\infty}\lambda^n\frac{(x-t)^n}{n!}$$

$$=1+\frac{\lambda(x-t)}{1!}+\frac{\lambda^2(x-t)^2}{2!}+\frac{\lambda^3(x-t)}{3!}+......$$

$$R(x,t,\lambda)=e^{\lambda(x-t)}$$

Example (2): Find the resolvent kernel of $k(x,t)=e^{x^2-t^2}$

Solution: - Iterated kernel is $k_1\left(x,t\right)=k\left(x,t\right)=e^{x^2-t^2}$

$$k_{n+1}\left(x,t\right)=\int_t^x k\left(x,z\right)k_n\left(z,t\right)dz$$

$$k_2\left(x,t\right)=\int_t^x k\left(x,z\right)k_1\left(z,t\right)dz$$

$$=\int_t^x e^{x^2-z^2}e^{z^2-t^2}dz=\int_t^x e^{x^2-t^2}dz=e^{x^2-t^2}\left(x-t\right)$$

$$k_3\left(x,t\right)=\int_t^x k\left(x,z\right)k_2\left(z,t\right)dz$$

$$=\int_t^x e^{x^2-z^2}e^{z^2-t^2}\left(z-t\right)dz$$

$$=\int_t^x e^{x^2-t^2}\left(z-t\right)dz=e^{x^2-t^2}\frac{\left(x-t\right)^2}{2!}$$

Similarly

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$$k_4(x,t)=e^{x^2-t^2}\frac{(x-t)^3}{3!}$$

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$$k_n(x,t)=e^{x^2-t^2}\frac{(x-t)^{n-1}}{(n-1)!}$$

$$k_{n+1}=(x,t)=e^{x^2-t^2}\frac{(x-t)^n}{n!},\ \mathrm{n}=0,1,2,...$$

By definition of resolvent kernel

$$R(x,t,\lambda)=\sum_{v=0}^{\infty}\lambda^v k_{v+1}(x,t)=\sum_{v=0}^{\infty}\lambda^v\frac{(x-t)^v}{v!}e^{x^2-t^2}$$

$$=e^{x^2-t^2}\sum_{v=0}^{\infty}\frac{(\lambda(x-t))^v}{v!}$$

$$\therefore\ R(x,t,\lambda)=e^{x^2-t^2}e^{\lambda(x-t)}$$

Example (3):- with aid of resolvent kernel find the solution of integral equation

$$\phi(x)=e^{x^2}+\int_0^x e^{x^2-t^2}\phi(t)dt \qquad (1)$$

Solution:-

$$k(x,t)=e^{x^2-t^2}$$

Now to find the resolvent kernel of equation (1).

(Write above solution.)

$$\therefore R(x,t,\lambda)=e^{x^2-t^2}e^{\lambda(x-t)}$$

The solution of volterra integral equation of second kind

$$\phi(x)=f(x)+\lambda\int_0^x R(x,t,\lambda)f(t)dt$$

$$f(x)=e^{x^2},R(x,t,\lambda)=e^{x^2-t^2}e^{\lambda(x-t)},\lambda=1$$

$$R(x,t,\lambda)=e^{x^2-t^2}e^{(x-t)}$$

The solution of integral equation (1) is

$$\phi(x) = f(x) + \lambda\int_0^x R(x,t,1) f(t) dt$$

$$= e^{x^2} + \int_0^x e^{x^2-t^2} e^{x-t} e^{t^2} dt$$

$$= e^{x^2} + e^{x^2} e^x \int_0^x e^{-t} dt = e^{x^2} + e^{x^2+x}\left(-\left(e^{-x}-1\right)\right)$$

$$= e^{x^2} - e^{x^2} + e^{x^2+x} = e^{x^2+x}$$

is the solution of (1).

Solution of Volterra Integral Equation and Some Application

4.1 The Method of Successive Approximation

Suppose the volterra-type integral equation of the second type be

$$\phi(x) = f(x) + \lambda\int_0^x k(x,t)\phi(t)dt \qquad (1)$$

Assume that $f(x)$ is continuous in [0, a] and the kernel $k(x,t)$ is continuous for $0 \le x \le a$, $0 \le t \le x$. Take some function $\phi_0(x)$ continuous in [0, a] put the function $\phi_0(x)$ into the R.H.S of (1) in place of $\phi(x)$ .Then,

$$\phi_1(x) = f(x) + \lambda\int_0^x k(x,t)\phi_0(t)dt \qquad (2)$$

This function $\phi_1(x)$ is also continuous in the interval [0, a]

Similarly take same function $\phi_1(x)$ by putting the $\phi_1(x)$ in R.H.S of (1) in place of $\phi(x)$, then

$$\phi_2(x) = f(x) + \lambda\int_0^x k(x,t)\phi_1(t)dt \qquad \ldots\ldots \quad (3)$$

This function $\phi_2(x)$ is also continuous in the interval [0, a], continuing in this process, we obtain a sequence of function.

$$\phi_0(x), \phi_1(x), \phi_2(x) \ldots\ldots \phi_n(x) \quad \text{where} \quad \phi_n(x) = f(x) + \lambda\int_0^x k(x,t)\phi_{n-1}(t)dt$$

Under the assumptions with respect to $f(x)$ and $k(x,t)$ the sequence $\{\phi_n(x)\}$ converges as $n \to \infty$ to the solution $\phi(x)$ of the integral equation (1).

In particular if for $\phi_0(x)$ we take $f(x)$, then $\phi_n(x)$ will be partial sums of the series

$$\phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) \ldots\ldots + \lambda^n\phi_n(x) = f(x) +$$
$$\lambda\int_0^x k(x,t)\left[\phi_0(\mathrm{t}) + \lambda\phi_1(\mathrm{t}) + \ldots\ldots + \lambda^n\phi_n(\mathrm{t}) + \ldots\right]dt$$

i.e.,

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$$\phi_0(x) = f(x)$$

$$\phi_1(x) = \int_0^x k(x,t)\phi_1(t)dt$$

$$\vdots$$

$$\vdots$$

$$\phi_n(x) = \int_0^x k(x,t)\phi_{n-1}(t)dt$$

$\therefore$ The partial sum of L.H.S of series (4) is the solution of the Integral Equation (1).

The zero approximation $\phi_0(x)$ can lead to a rapid converge of the sequence $\{\phi_n(x)\}$ to the solution of integral equation.

Example (1): Using the method of successive approximations. Solve the Integral equation

$$\phi(x) = x - \int_0^x (x-t)\phi(\mathrm{t})\,\mathrm{dt};\ \phi_0(x) = 0 \qquad (1)$$

Solution:

$$\phi_n(x) = f(x) + \lambda\int_0^x \mathrm{k}(x,t)\phi_{n-1}(t)dt \qquad (3)$$

Since $\phi_0(x) = 0,$ from (1) & (2)

$$\phi_1(x) = x - \int_0^x (x-t)\phi_0(t)dt = x$$

$$\phi_2(x) = x - \int_0^x (x-t)(t)$$

$$= x - x\int_0^x t + \int_0^x t^2 dt = x - x.\frac{x^2}{2} + \frac{x^3}{3}$$

$$= \frac{6x - 3x^3 + 2x^3}{6} = \frac{6x - x^3}{6}$$

$$= x - \frac{x^3}{3!}$$

$$\phi_3(x) = x - \int_0^x (x-t)(t - \frac{t^3}{6})dt$$

$$= x - x\int_0^x (t - \frac{t^3}{6})dt + \int_0^x (t^2 - \frac{t^4}{6})dt$$

$$= x - x\left[\frac{t^2}{2} - \frac{t^4}{24}\right]_0^x + \left[\frac{t^3}{3} - \frac{t^5}{30}\right]_0^x$$

$$= x - \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^3}{3} - \frac{x^5}{30}$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\phi_4(x) = x - \int_0^x (x-t)(t - \frac{t^3}{3!} + \frac{t^5}{5!})dt = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$$

$$\therefore \phi_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

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Hence $\phi_n(x)$ is the n^{th} partial sum of the series

$\Rightarrow \phi_n(x) \to \mathrm{sinx}$ as $n \to \infty$

$\therefore \phi(x) = \sin x$ is the solution of integral equation (1)

OR

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$$\phi_0(x) = f(x) = x$$

$$\phi_1(x) = \int_0^x (x-t)t\,dt = \frac{x^3}{3!}$$

$$\phi_2(x) = \int_0^x (x-t)(-\frac{t^3}{3!}) = \frac{x^5}{5!}$$

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$$\phi_0 + \lambda\phi_1(x) + \lambda^2\phi_2(x) + ... = x - 1\,\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sin x.$$

Convolution type integral equations:

The volterra integral equation of second kind $\phi(x) = f(x) + \int_0^x k(x,t)\phi(t)dt$ (1)

Where the kernel $k(x,t)$ is dependent solely on the difference $x-t$ is called volterra integral equation of convolution type.

Note: Let $\phi_1(x)\,\&\,\phi_2(x)$ be two functions defined on $x \geq 0$. The convolution of these two functions is

$$\phi_3(x) = \int_0^x \phi_1(x-t)\phi_2(t)dt = \int_0^x \phi_2(x-t)\phi_1(t) = \phi_1.\phi_2 \qquad (2)$$

This function $\phi_3(x)$, defined for $x \geq 0$ is also continuous function. If $\phi_1(x)\,\&\,\phi_2(x)$ are original functions for the Laplace transformations

i.e.., $L(\phi_1(x)) = \overline{\phi_1}(p), L(\phi_2(x)) = \overline{\phi_2}(p)$

Then $L(\phi_3) = L(\phi_1(x)).L(\phi_2(x))$

$$\overline{\phi_3}(p) = \overline{\phi_1}(p).\overline{\phi_2}(p)$$

From the volterra integral equation (1), Taking Laplace Transform on both sides of (1) by using convolution theorem

$$L\{\phi(x)\} = L\{f(x)\} + L\{K(x)\}.L\{\phi(x)\}$$

$$\bar{\phi}(P) = \bar{f}(P) + \bar{K}(P)\bar{\phi}(P)$$

$$\Rightarrow \bar{\phi}(P)\left[1 - \bar{K}(P)\right] = \bar{f}(P)$$

Let

$$L\{\phi(x)\} = \bar{\phi}(P)$$
$$L\{K(x)\} = \bar{K}(P)$$
$$L\{f(x)\} = \bar{f}(P)$$
$$\left(\bar{K}(P) \neq 0\right)$$
$$\Rightarrow \bar{\phi}(P) = \frac{\bar{f}(P)}{1-\bar{K}(P)} = A(P) \quad (Say) \qquad (4)$$

Then the original function $\phi(x)$ of $\phi(P)$ is the solution of the given integral equation (1)

i.e.., $\phi(x) = L^{-1}\{\bar{\phi}(P)\} = L^{-1}\{A(P)\}$

Example (1):

Solve the integral equation $\phi(x) = \sin x + 2\int_0^x \cos(x-t)\phi(t)dt$ (1)

Solution: $f(x) = \sin x, \lambda = 2, K(x,t) = \cos(x-t) = K(x-t)$

.. Integral equation (1) of convolution type

Taking Laplace Transformation on both sides (1) by using convolution theorem

$$L\{\phi(x)\} = L\{f(x)\} + 2L\{k(x)\}L\{\phi(x)\}$$
$$= L\{\sin x\} + 2L\{\cos x\}L\{\phi(x)\}$$
$$k(x-t) = \cos(x-t)$$
$$k(x) = \cos x$$
$$L(k(x)) = \frac{P}{P^2+1}$$
$$L(\sin x) = \frac{1}{P^2+1}$$
$$L(\phi(x)) = \bar{\phi}(P)$$
$$\bar{\phi}(P) = \frac{1}{P^2+1} + 2.\left\{\frac{P}{P^2+1}\right\}\bar{\phi}(P)$$
$$\bar{\phi}(P)\left[1-\frac{2P}{P^2+1}\right] = \frac{1}{P^2+1}$$

$$\bar{\phi}(P).\frac{(P-1)^2}{P^2+1}=\frac{1}{P^2+1}\Rightarrow\bar{\phi}(P)=\frac{1}{(P-1)^2}$$

$\therefore \phi(x)=L^{-1}\left[\bar{\phi}(P)\right]=L^{-1}\left[\bar{\phi}(P)\right]=L^{-1}\left[\frac{1}{(P-1)^2}\right]=e^x.x$ is the solution of Integral Equation (1)

$$\therefore \phi(x,t)=e^{(x-t)}(x-t)$$

Reduce the boundary value problem in to volterra integral equation:-

Example (1): Froman integral equation corresponding to the differential equation

$y'''+xy''+(x^2-x)y=xe^x+1 \qquad (1)$ with initial condition $y(0)=0.$

$y'(0)=1,\ y''(0)=0 \qquad (2)$

Solution:

We have

$$y'''=\frac{d^3y}{dx^3}=\phi(x) \qquad (3)$$

Then

$$\frac{d^2y}{dx^2}=\int_0^x\phi(t)dt+y''(0)=\int_0^x\phi(t)dt$$

$$\therefore \frac{dy}{dx}=\int_0^x(x-t)\phi(t)\,\mathrm{dt}+\mathrm{c}_2\,x++c_1=\int_0^x(x-t)\phi(\mathrm{t})\,\mathrm{dt}+1$$

$$y=\int_0^x\frac{(x-t)^2}{2}\phi(t)dt+c_2\frac{x^2}{2}+c_1x+c_0$$

$$=\int_0^x\frac{(x-t)^2}{2}\phi(t)dt+x$$

Substituting $y,\ y'',\ y'''$ in (1) we get

$$\phi(x)+x\int_0^x\phi(t)dt+(x^2-x)\left[\int_0^x\frac{(x-t)^2}{2}\phi(t)dt+x\right]=xe^x+1$$

$$\Rightarrow\phi(x)+x(x^2-x)+\int_0^x\phi(t)\left[x+\frac{(x^2-x)(\mathrm{x-t})^2}{2}\right]dt=xe^x+1$$

$$\phi(x)=xe^x+1-x(x^2-x)-\int_0^x\frac{2x+(x^2-x)(x-t)^2}{2}\phi(t)dt \qquad (5)$$

It is in the form $\phi(x)=f(x)+\lambda\int_0^x k(x,t)\phi(t)dt$ volterra integral equation of second kind.

where

$$f(x)=-x(x^2-x)+xe^x+1$$

$$k(x,t)=\frac{2x+(x-t)^2(x^2-x)}{2}\ ,\lambda=-1$$

Example 2: Convert the differential equation $\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-3y=0$ (1) with initial conditions $y(0)=1, y'(0)=0$ (2) to an integral equation.

Solution: Put $\frac{d^2y}{dx^2}=\phi(x)$ (3)

Integrating with respect to '*x*'

$$\int_0^x\frac{d^2y}{dx^2}dx=\int_0^x\phi(x)dx$$

$$y'(x)-y'(0)=\int_0^x\phi(x)dx=\int_0^x\phi(t)dt$$

$$y'(0)=\int_0^x\phi(t)dt \qquad (4)$$

again integrating with respect to ' x '

$$y(x)-y(0)=\int_0^x\phi(t)dt^2$$

$$y(x)-1=\int_0^x(x-t)\phi(t)dt$$

$$y(x)=1+\int_0^x(x-t)\phi(t)dt \qquad (5)$$

Substituting (3), (4), (5) in (1) we get

$$\phi(x)-2x\int_0^x\phi(t)dt-3\left[1+\int_0^x(x-t)\phi(t)dt\right]=0$$

$$\phi(x)=3+\int_0^x\left[2x+3(x-t)\right]\phi(t)dt$$

$\phi(x)=3+\int_0^x\left[5x-3t\right]\phi(t)dt$ is the volterra Integral equation of second Kind.

$$f(x)=3,k(x,t)=5x-3t,\lambda=-1.$$

Example 3: Convert the differential equation $\frac{d^2\phi}{dx^2}-2x\frac{d\phi}{dx}-3\phi=0$ (1) with initial condition $\phi(0)=1,\phi'(0)=0$ (2) to volterra Integral equation second kind, conversely derived the original differential equation with the initial conditions from the obtained integral equation.

Solution: $\frac{d^2\phi}{dx^2}=2x\frac{d\phi}{dx}+3\phi$

Integrating on both sides

$$\left.\frac{d\phi}{dx}\right]_0^x=\int_0^x 2x\phi'(x)dx+3\int_0^x\phi(x)dx$$

$$\phi'(x)-\phi'(0)=\phi'(x)-0=2\left\{x\phi(x)\right\}=\int_0^x\phi(x)dx+3\int_0^x\phi(x)dx$$

$$\phi'(x)-\phi'(0)=2\int_0^x x\phi(x)=\int_0^x\phi(x)dx+3\int_0^x\phi(x)dx$$

$$\phi'(x)=2x\phi(x)+\int_0^x\phi(x)dx \qquad (3)$$

On integrating we get

$$\phi(x)-\phi(0)=2\int_0^x x\phi(x)+\int_0^x\phi(x)dx^2$$

$$\phi(x)-1=2\int_0^x t\phi(t)+\int_0^x\phi(t)dt^2$$

$$\phi(x)=1+2\int_0^x t\phi(t)dt+\int_0^x\frac{(x-t)}{1!}\phi(t)dt$$

$$\phi(x)=1+2\int_0^x t\phi(t)dt+\int_0^x(x-t)\phi(t)dt$$

$$\phi(x)=1+\int_0^x(2t+x.t)\phi(t)dt=1+\int_0^x(t+x)\phi(t)dt \qquad (4)$$

Isthe volterra Integral equation of second kind where $k(x,t)=x+t,\ f(x)=1\ \&\ \lambda=1$

4.3 Convert the differential equation $\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-3y=0$ (1) with initial conditions $y(0)=1, y'(0)=0$ to an integral equation (2)

Solution:

Put $\frac{d^2y}{dx^2}=\phi(x)$ (3)

Integrating with respect to '*x*'

$$\int_0^x \frac{d^2y}{dx^2}dx=\int_0^x \phi(x)dx$$

$$y'(x)-y'(0)=\int_0^x \phi(x)dx=\int_0^x \phi(t)dt$$

$$y'(x)=\int_0^x \phi(t)dt \qquad (4)$$

again integrating with respect to $'x'$

$$y(x)-y(0)==\int_0^x \phi(t)dt^2$$

$$y(x)-1=\int_0^x \frac{(x-t)}{1}\phi(t)dt$$

$$y(x)=1+\int_0^x (x-t)\phi(t)dt \qquad (5)$$

substituting (3),(4),&(5) in (1) we get

$$\phi(x)-2x\int_0^x \phi(t)dt-3\left[1+\int_0^x (x-t)\phi(t)dt\right]=0$$

$$\phi(x)=3+\int_0^x [2x+3(x-t)]\phi(t)dt$$

$\phi(x)=3+\int_0^x [(5x-3t)]\phi(t)dt$ is the volterra integral equation of second kind

$f(x)=3, k(x,t)=5x-3t, \lambda=1$

Example (4) : Convert the differential equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-3y=0 \quad (1) \text{ with initial conditions}$$

$$y(0)=1, y'(0)=0 \text{ to an integral equation} \quad (2)$$

Solution:

Put $\frac{d^2y}{dx^2}=\phi(x)$ (3)

Integrating with respect to 'x'

$$\int_0^x \frac{d^2y}{dx^2}dx=\int_0^x \phi(x)dx$$

$$y'(x)-y'(0)=\int_0^x \phi(x)dx=\int_0^x \phi(t)dt$$

$$y'(x)=\int_0^x \phi(t)dt \quad (4)$$

again integrating with respect to 'x'

$$y(x)-y(0)==\int_0^x \phi(t)dt^2$$

$$y(x)-1=\int_0^x \frac{(x-t)}{1}\phi(t)dt$$

$$y(x)=1+\int_0^x (x-t)\phi(t)dt \quad (5)$$

substituting (3),(4),&(5) in (1) we get

$$\phi(x)-2x\int_0^x \phi(t)dt-3\left[1+\int_0^x (x-t)\phi(t)dt\right]=0$$

$$\phi(x)=3+\int_0^x [2x+3(x-t)]\phi(t)dt$$

$\phi(x)=3+\int_0^x [(5x-3t)]\phi(t)dt$ is the volterra integral equation of second kind

$$f(x)=3, k(x,t)=5x-3t, \lambda=1$$

4.4 Volterra Integral Equation of First Kind:

The Volterra Integral Equation of first kind is in the form $f(x)=\int_0^x k(x,t)\phi(t)dt$

If the Volterra Integral Equation of first kind is $\int_0^x k(x,t)\phi(t)dt = f(x),\ f(0)=0 \quad (1)$ where $\phi(x)$ is unknown function.

Suppose $k(x,t), \frac{\partial}{\partial x}k(x,t), f(x)\ \&\ f'(x)$ are continuous $0 \le x \le a\ \&\ 0 \le t \le x$

Differentiating both sides of (1) with respect to $'x'$. Then

$$k(x,x)\phi(x) + \int_0^x \frac{\partial}{\partial x}k(x,t)\phi(t)dt = f'(x) \tag{2}$$

$$\left[\frac{d}{dx}\int_{P(x)}^{Q(x)} F(x,t)dt\right] = \int_{P(x)}^{Q(x)} \frac{\partial}{\partial x}F(x,t)dt + \frac{\partial}{\partial x}Q(x)F(x,Q(x) - \frac{\partial}{\partial x}P(x)F(x,P(x))$$

Any continuous solution $\phi(x)$ of eq (1) for $0 \le x \le a$ satisfies (2) as well. Conversely any continuous solution of eq (2), for $0 \le x \le a$ satisfies eq (1) too.

Case (i): if $k(x,x)$ doesnot vanish at any point on $[0,a]$ then eq (2) can be written as

$$\phi(x) = \frac{f'(x)}{k(x,x)} - \frac{\int_0^x \frac{\partial}{\partial x}(k(x,x))\phi(t)dt}{k(x,x)} \tag{3}$$

The eqn (3) is in the form $\phi(x) = f(x) + \lambda\int_0^x k(x,t)\phi(t)dt$ i.e.., the eqn reduced in the form of volterra integral equation of second kind.

The solution of (3) is find some usual methods

$\therefore$ The solution of (3) is also solution of (1)

Case (ii): If $k(x,x) = 0$

The eqn (2) is again first kind. Then differentiating again (2) with respect to $'x'$ repeatedly to reduce into second kind.

Example (1): Solve the Integral Equation of first kind by reducing to the second kind $\int_0^x e^{x-t}\phi(t)dt = \sin x$ (1)

Solution: $K(x-t) = e^{x-t},\ f(x) = \sin x,\ f(0) = 0$

Differentiating (1) with respect to $'x'$ we get

$$k(x,x)\phi(x) + \int_0^x \frac{\partial}{\partial x}k(x,t)\phi(t)dt = f'(x)$$

$$e^{x-x}\phi(x)+\int_0^x \frac{\partial}{\partial x} e^{x-t}\phi(t)=\left(\sin x\right)'$$

$$\Rightarrow 1.\phi(x)+\int_0^x \frac{\partial e^x}{\partial x}.e^{-t}\phi(t)dt=\cos x$$

$$=\phi(x)+\int_0^x e^x.e^{-t}\phi(t)dt=\cos x \quad \ldots\ldots \quad (2)$$

Here $k(x,x)\neq 0$

$\therefore$ The eqn(2) is reduced in the form

$$\phi(x)=\cos x-\int_0^x e^{x-t}\phi(t)dt \qquad (3)$$

Is in the form volterra integral equation of second kind.

The resolvent kernel for $k(x,t)=e^{x-t}$ is $R(x,t,\lambda)=e^{(\lambda+1)(x-t)}=1$

$\therefore$ The solution of (3)

$$\phi(x)=f(x)+\lambda\int_0^x R(x,t,\lambda)f(t)dt$$

$$=\cos x-\int_0^x 1.\cos t\ dt=\cos x-(\sin t)_0^x$$

$$=\cos x-\sin x$$

$\therefore \phi(x)=\cos x-\sin x$ is the solution of (1).

CONCLUSION:

The study concludes the solution of integral transformation problems and its application. The study also presents the volterra integral transformation problem and kernel problems and there solution and applications. The study is to obtain results on the differentiability properties of solutions of nonlinear Volterra integral equations of the second kind with convolution kernels $a(t - s)$. It is assumed that $a(t)$ is continuous for t> 0 and integrable at the origin although $a(t)$ may become unbounded at $t = 0$.

The kernel trick avoids the explicit mapping that is needed to get linear learning algorithms to learn a nonlinear function or decision boundary. For all X and X' in the input space χ, certain functions $k(X,X')$ can be expressed as an inner product in another space v. The function $k:\chi\times\chi\rightarrow R$ is often referred to as a kernel or a kernel function. The word "kernel" is used in mathematics to denote a weighting function for a weighted sum or integral.

Certain problems in machine learning have additional structure than an arbitrary weighting function k. The computation is made much simpler if the kernel can be written in the form of a "feature map" $\varphi:\chi\rightarrow v$ which satisfies

$$k(x,x')=\langle\varphi(x),\varphi(x')\rangle v$$

Theoretically, a Gram matrix $K\in R^{n\times n}$ with respect to $\{X_1,\ldots\ldots X_n\}$ (sometimesalso called a "kernel matrix"), where $K = (k(X_i, X_j))_{ij}$ positive semi-definite(PSD). Empirically, for machine learning heuristics, choices of a function k that do not satisfy Mercer's condition may still perform reasonably if k at least approximates the intuitive idea of similarity. Regardless of whether k is a Mercer kernel, k may still be referred to as a "kernel". If the kernel function k is also a covariance function as used in Gaussian processes, then the Gram matrix K can also be called a covariance matrix.

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