International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 03
March 2017

Comparison between Fourier and Laplace Transform

Own Ghareeb Kareem
Master of Science (Applied Mathematics) Department Of Mathematics University College of Science, Osmania University, India

Abstract:

To present the laplace and fourier transformations with their applications. To present the comparison analysis between laplace and fourier transformation. This part of the course introduces two extremely powerful methods to solving differential equations: the Fourier and the Laplace transforms. Beside its practical use, the Fourier transform is also of fundamental importance in quantum mechanics, providing the correspondence between the position and momentum representations of the Heisenberg commutation relations.

Keywords: Differential Equations, Fourier Transformations, Heisenberg Commutation, Laplace Transformations.

INTEGRAL TRANFORMATION

The precursor of the transforms were the Fourier series to express functions in finite intervals. Later the Fourier transform was developed to remove the requirement of finite intervals.

Using the Fourier series, just about any practical function of time (the voltage across the terminals of an electronic device for example) can be represented as a sum of sines and cosines, each suitably scaled (multiplied by a constant factor), shifted (advanced or retarded in time) and "squeezed" or "stretched" (increasing or decreasing the frequency). The sines and cosines in the Fourier series are an example of an orthonormal basis (*(Sneddon, I. N. (1972))*).

In mathematics, an **integral transform** is any transform *T* of the following form:

$$(Tf)(u) = \int_{t_1}^{t_2} K(t,u)\, f(t)\, dt$$

The input of this transform is a function *f*, and the output is another function *Tf*. An integral transform is a particular kind of mathematical operator.

There are numerous useful integral transforms. Each is specified by a choice of the function *K* of two variables, the **kernel function** or **nucleus** of the transform. Some kernels have an associated *inverse kernel* $K^{-1}(u, t)$ which (roughly speaking) yields an inverse transform:

$$f(t) = \int_{u_1}^{u_2} K^{-1}(u,t)\, (Tf)(u)\, du$$

A *symmetric kernel* is one that is unchanged when the two variables are permuted.

EXAMPLE:

As an example of an application of integral transforms, consider the Laplace transform. This is a technique that maps differential or integro-differential equations in the "time" domain into polynomial equations in what is termed the "complex frequency" domain. (Complex frequency is similar to actual, physical frequency but rather more general. Specifically, the imaginary component ω of the complex frequency $s = -\sigma + i\omega$ corresponds to the usual concept of frequency, *viz.*, the rate at which a sinusoid cycles, whereas the real component σ of the complex frequency corresponds to the degree of "damping".) The equation cast in terms of complex frequency is readily solved in the complex frequency domain (roots of the polynomial equations in the complex frequency domain correspond to eigenvalues in the time domain), leading to a "solution" formulated in the frequency domain. Employing the inverse transform, *i.e.*, the inverse procedure of the original Laplace transform, one obtains a time-domain solution. In this example, polynomials in the complex frequency domain (typically occurring in the denominator) correspond to power series in the time domain, while axial shifts in the complex frequency domain correspond to damping by decaying exponentials in the time domain (*Tranter, C. J. (1951))*.

The Laplace transform finds wide application in physics and particularly in electrical engineering, where the characteristic equations that describe the behavior of an electric circuit in the complex frequency domain correspond to linear combinations of exponentially damped, scaled, and time-shifted sinusoids in the time domain. Other integral transforms find special applicability within other scientific and mathematical disciplines (*(Sneddon, I. N. (1972)))*.

Another usage example is the kernel in path integral:

$$\psi(x,t) = \int_{-\infty}^{\infty} \psi(x',t') K(x,t;x',t') dx'.$$

This state that the total amplitude to arrive at (x,t)[that is, $\psi(x,t)$] is the sum, or the integral, over all possible value of x' of the total amplitude to arrive at the point (x',t')[that is, $\psi(x',t')$] multiplied by the amplitude to go from x' to x [that is, $K(x,t;x',t')$.[1] It is often referred to as the propagator of a given system. This (physics) kernel is the kernel of integral transform. However, for each quantum system, there is a different kernel.

Laplace–Stieltjes transform

The (unilateral) Laplace–Stieltjes transform of a function $g : \mathbf{R} \rightarrow \mathbf{R}$ is defined by the Lebesgue–Stieltjes integral

The function *g* is assumed to be of bounded variation. If *g* is the antiderivative of *f*:

$$g(x) = \int_0^x f(t)\,dt$$

then the Laplace–Stieltjes transform of *g* and the Laplace transform of *f* coincide. In general, the Laplace–Stieltjes transform is the Laplace transform of the Stieltjes measure associated to *g*. So in practice, the only distinction between the two transforms is that the Laplace transform is thought of as operating on the density function of the measure, whereas the Laplace–Stieltjes transform is thought of as operating on its cumulative distribution function.

Fourier transform

The continuous Fourier transform is equivalent to evaluating the bilateral Laplace transform with imaginary argument $s = i\omega$ or $s = 2\pi fi$:

$$\begin{aligned}\hat{f}(\omega) &= \mathcal{F}\{f(t)\} \\ &= \mathcal{L}\{f(t)\}|_{s=i\omega} = F(s)|_{s=i\omega} \\ &= \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt.\end{aligned}$$

This definition of the Fourier transform requires a prefactor of $1/2\pi$ on the reverse Fourier transform. This relationship between the Laplace and Fourier transforms is often used to determine the frequency spectrum of a signal or dynamical system. The above relation is valid as stated if and only if the region of convergence (ROC) of $F(s)$ contains the imaginary axis, $\sigma = 0$. For example, the function $f(t) = \cos(\omega_0 t)$ has a Laplace transform $F(s) = s/(s^2 + \omega_0^2)$ whose ROC is $\mathrm{Re}(s) > 0$. As $s = i\omega$ is a pole of $F(s)$, substituting $s = i\omega$ in $F(s)$ does not yield the Fourier transform of $f(t)u(t)$, which is proportional to the Dirac delta-function $\delta(\omega-\omega_0)$.

However, a relation of the form

$$\lim_{\sigma \to 0^+} F(\sigma + i\omega) = \hat{f}(\omega)$$

holds under much weaker conditions. For instance, this holds for the above example provided that the limit is understood as a weak limit of measures (see vague topology). General conditions relating the limit of the Laplace transform of a function on the boundary to the Fourier transform take the form of Paley-Wiener theorems.

Fourier transform to Laplace transform

The condition of the existence of the Fourier transform of $f(t)$ is the convergence of the integral

$$\int_{-\infty}^{\infty} |f(t)|\, dt \qquad (1.1)$$

In many physical problems, $f(t)$ may be assumed to be identically equal to zero when $t < 0$, and $f(t)$ does not make the integral (1.1) convergent, we consider instead of the function

$$g(t) = \begin{cases} e^{-at} f(t), & t > 0 \\ 0, & t < 0 \end{cases}$$

where a is a positive constant of such a nature that the integral

$$\int_{-\infty}^{\infty} |e^{-at} f(t)|\, dt = \int_{0}^{\infty} |e^{-at} f(t)|\, dt \qquad (1.2)$$

does converge, i.e., $|f(t)| < M e^{at}$ for large value of t, $f(t)$ is called an exponential order function. Then the Fourier transform of $g(t)$ is

$$F[g(t)] \square \int_0^{\infty} e^{-at} f(t) e^{-i\omega t} dt = \int_0^{\infty} f(t) e^{-(a+i\omega)t} dt$$

Inverse Fourier transform is

$$e^{-at} f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\ [g(t)]e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_0^{\infty} f(t)e^{-(a+i\omega)t}dt\right]e^{i\omega t}d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_0^{\infty} f(t)e^{-(a+i\omega)t}dt\right]e^{(a+i\omega)t}d\omega \qquad (1.3)$$

Let $s\ \ a + i$, equation (1.3) becomes

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[\int_0^{\infty} f(t)e^{-st}dt\right]e^{st}ds$$

Define

$$F(s) = \mathcal{L}\ [f(t)] = \int_0^{\infty} f(t)e^{-st}dt \qquad (1.4)$$

as the Laplace transform of $f(t)$, and the inverse Laplace transform is

$$f(t) = \mathcal{L}^{\square 1}[F(s)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}ds \qquad (1.5)$$

Theorem: The conditions for the existence of Laplace transform of $f(t)$ are

(1) $f(t)$ is piecewise continuous in every finite interval.

(2) $f(t)$ is an exponential order function.

$$\Gamma(a+1) = \int_0^{\infty} e^{-u}u^{(a+1)-1}du = \int_0^{\infty} e^{-u}u^{a}du = -(e^{-u}u^{a}\Big|_0^{\infty} - a\int_0^{\infty} e^{-u}u^{a-1}du) = a\int_0^{\infty} e^{-u}u^{a-1}du = a\Gamma(a)$$

$$\Gamma(1) = \int_0^{\infty} e^{-u}u^{1-1}du = \int_0^{\infty} e^{-u}du = -e^{-u}\Big|_0^{\infty} = 1$$

When n is a natural number, then

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2)$$
$$= \cdots = n(n-1)(n-2)\cdots\times 2\times 1\times\Gamma(1) = n!$$

$$\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-u}u^{\frac{1}{2}-1}du = \int_0^{\infty} e^{-u}u^{-\frac{1}{2}}du = 2\int_0^{\infty} e^{-u}d\sqrt{u} = 2\int_0^{\infty} e^{-x^2}dx = 2\sqrt{\int_0^{\infty} e^{-x^2}dx\cdot\int_0^{\infty} e^{-y^2}dy}$$

$$= 2\sqrt{\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy} = 2\sqrt{\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}rdrd\theta} = 2\sqrt{\frac{1}{2}\int_0^{\frac{\pi}{2}}(-e^{-r^2})_0^{\infty}d\theta} = 2\sqrt{\frac{1}{2}\int_0^{\frac{\pi}{2}}1d\theta}$$

$$= 2\sqrt{\frac{1}{2}\cdot\frac{\pi}{2}} = \sqrt{\pi}$$

COMPARISON BETWEEN FOURIER AND LAPLACE TRANSFORMSATION

First, lets take a look at the Fourier Transform (FT) of a CT signal, x(t),

$$X(\omega) = F[x(t)] = x(t)\ e^{-j\omega t}\ \ dt.$$

The FT transforms a time-domain signal into a frequency-domain signal, telling us how the signal's energy ("information") is spread across sinusoids of different frequencies. This property makes the FT invaluable in many signal processing tasks such as audio engineering and wireless communications (though, the applications of the FT are limitless). Now, remember that the FT had the following condition: the F[x(t)] only exists if

∞

$\int |x(t)|\ dt < \infty.$
$-\infty$

In other words, F[x(t)] exists only if the total energy of our signal is bounded, that is, the area underneath its curve is finite.

Why does the FT have this condition? Lets attempt to find the FT of the two unacceptable signals shown above. For the first case, we observe a unit step function,

$$x(t) = u(t),$$

which is defined to have a value of 1 for all positive t. So, attempting to calculate F[u(t)],

$$X(\omega) = F(u(t)) = \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, dt,$$

The integral ∞ is, technically, an improper integral. We will note that by setting a bound to infinity we are actually saying R that we will calcu~~late~~ this integral as the upper end approaches infinity in the limit. Remembering this fact we find

$$X(\omega) = -\int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, dt\ ,$$

Unfortunately for us, the term $LIM_{t\to\infty}\ e^{-j\omega t}$ doesn't converge to anything in the limit since the term

$$e^{-j\omega t} = {}^{1}{}_{2}\ [COS\ \omega t - j\ SIN\ \omega t]$$

and here, again, we see that for our stated condition α > 0, the integral diverges as the term $LIM_{t\to\infty}$ $e^{\alpha t}e^{-j\omega t}$ goes to infinity. And so, X(ω) does not exist. While the FT is useful for analyzing signals in terms of their frequency composition, many kinds of signals, such as the two we have just investigated, defy strict Fourier transformation (though general transforms can be arrived at.)

To analyze a more general set of functions, including functions which may not have FTs, we can use the Laplace transform (LT). The LT can be thought of as a generalized FT. The LT is defined as

$$X(\omega) = F(u(t)) = \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, dt,$$

where

$$s = \sigma + j\omega.$$

Where the FT of a signal, X(ω), was a function of only imaginary numbers, the LT of the same signal, X(s), is a function of complex numbers consisting for both real and imaginary components. We can view the FT as a special case of the LT for which σ = 0.

The inverse FT tells us that the function x(t) can be constructed from complex sinusoids which are denoted by the $e^{j\omega t}$ term. An example of one such complex sinusoid is given in Fig. 1. These complex sinusoids have a fixed amplitude across time. It is this fixed amplitude which prevents us from analyzing signals whose energy is not bounded. Simply put, the FT's complex sinusoids do not have enough explanatory power for these functions.

Now, let us contrast the FT's construction of x(t) with that of the LT. The inverse LT is defined as

$$x(t) = \frac{1}{2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} X(\omega)\, e^{(\sigma+j\omega)t}\ dt.$$

Rather than complex sinusoids of fixed amplitude, the LT defines x(t) in terms of complex sinusoids whose amplitudes exponentially grow or decay with respect to time, Fig. 2 gives an example of one

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 03
March 2017

such exponentially growing sinusoid. The additional exponential term $e^{\sigma t}$ gives the LT much more flexibility than the FT for representing signals whose energy is unbounded.

Now, lets use the LT to find the transformation of the unit step and exponential growth functions for which the FT does not exist. Note that we are using the unilateral LT, which integrates over $[0^-, \infty]$, rather than the bilateral LT which integrates over $[-\infty, \infty]$. We use the unilateral LT for analyzing casual signals, i.e. signals for which x(t) = 0, t < 0. In our case, both the unit step and exponential growth functions we analyze here are causal. First, the unit step, Here we see the extra leverage that the LT gives us over the FT. We now have an additional term, σ which we can use to force this integral to converge. Specifically,

$$\lim_{t\to\infty} e^{-\sigma t} e^{-j\omega t} = 0, \quad \sigma > 0.$$

Note, however, that this convergence only occurs for certain values of σ, in this case σ > 0. Therefore,

$$X(s) = L[u(t)] = \frac{1}{\sigma + j\omega} \overset{[-1]}{=} \frac{1}{s}, \quad \sigma > 0.$$

We make the note here that the region in which the LT exists (in this case, σ > 0) is called the Region of Convergence (ROC) for the LT. Each different LT has a different ROC in which the LT exists. Finally, we see that the step function, which did not have a FT due to its infinite energy, does indeed have an LT (though only within the ROC).

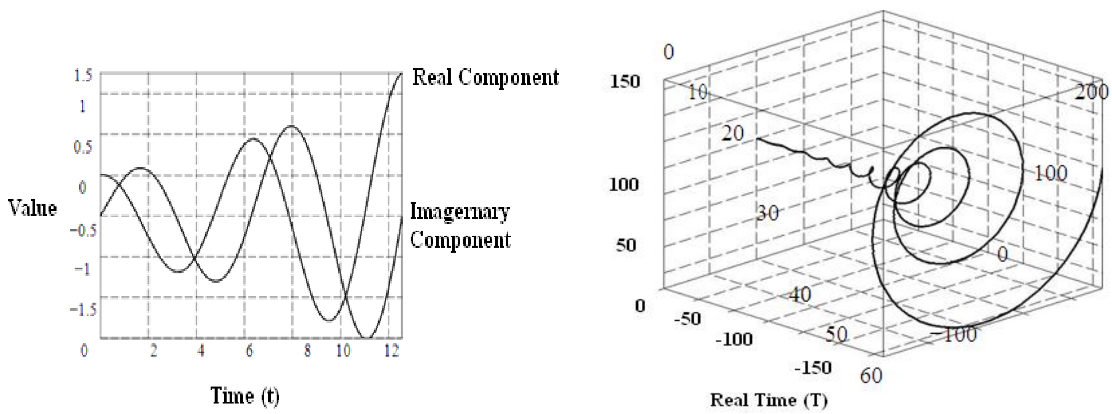

Figure 2: Complex sinusoid $e^{(0.1-j)t}$.

REGION OF CONVERGENCE (ROC)

Whether the Laplace transform $X(s)$ of a signal $x(t)$ exists or not depends on the complex variable s as well as the signal itself. All complex values of s for which the integral in the definition converges form a *region of convergence (ROC)* in the s-plane. $X(s)$ exists if and only if the argument s is inside the ROC. As the imaginary part $\omega = Im[s]$ of the complex variable $s = \sigma + j\omega$ has no effect in terms of the convergence, the ROC is determined solely by the real part $\sigma = Re[s]$.

Example 1: The Laplace transform of $x(t) = e^{-at}u(t)$ is:

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-at}e^{-(\sigma+j\omega)t}dt$$
$$= -\frac{1}{a+\sigma+j\omega}\, e^{-(a+\sigma+j\omega)t}\Big|_0^\infty$$

For this integral to converge, we need to have

$$a+\sigma > 0 \quad \text{or} \quad \sigma = Re[s] > -a$$

and the Laplace transform is

$$X(s) = \frac{1}{(\sigma+a)+j\omega} = \frac{1}{s+a}$$

As a special case where $a = 0$, $x(t) = u(t)$ and we have

$$\mathcal{L}[u(t)] = \frac{1}{s}, \qquad \sigma = Re[s] > 0$$

Example 2: The Laplace transform of a signal $x(t) = -e^{-at}u(-t)$ is:

$$X(s) = -\int_{-\infty}^{0} e^{-at}e^{-st}dt = -\int_{-\infty}^{0} e^{-(a+\sigma+j\omega)t}dt = \frac{1}{a+\sigma+j\omega}\, e^{-(a+\sigma+j\omega)t}\Big|_{-\infty}^{0}$$

Only when $a+\sigma < 0 \quad \text{or} \quad \sigma = Re[s] < -a$

will the integral converge, and Laplace transform $X(s)$ is

$$X(s) = \frac{1}{a+\sigma+j\omega} = \frac{1}{a+s}$$

Again as a special case when $a = 0$, $x(t) = -u(-t)$ we have

$$\mathcal{L}[-u(-t)] = \frac{1}{s}, \qquad \sigma = Re[s] < 0$$

Comparing the two examples above we see that two different signals may have identical Laplace transform $X(s)$, but different ROC. In the first case above, the ROC is $Re[s] > 0$, and in the second case, the ROC is $Re[s] < 0$. To determine the time signal $x(t)$ by the inverse Laplace transform, we need the ROC as well as $X(s)$. Now we turn our attention to the exponential growth function, And, once again, we see that the LT gives us to ability to force this integral to converge by setting σ−α > 0, that is, σ > α. Finally,

Before we move on, let us make a few observations. First, we see that the LT of the unit step function and of the exponential growth function are related. Let us, instead of considering just an exponentially growing function, allow α to take on any value, positive or negative. We now have a function which

- Grows when $\alpha > 0$,
- Is constant when $\alpha = 0$,
- Decays when $\alpha < 0$.

When we set $\alpha = 0$, we see that $e^{\alpha t}u(t) \rightarrow u(t)$. Thus,

$$L[e^{(\alpha=0)t}u(t)] = \frac{1}{s-0} = 1/s = L[u(t)].$$

Now, lets look at the ROC for the LT of our general, causal, exponential function, $x(t) = e^{\alpha t}u(t)$. The ROC for differing values of α are given in Figs. 3–5. In the case that $\alpha = 0$ or $\alpha > 0$, i.e. a unit step and an exponential growth function, the FT does not exist.

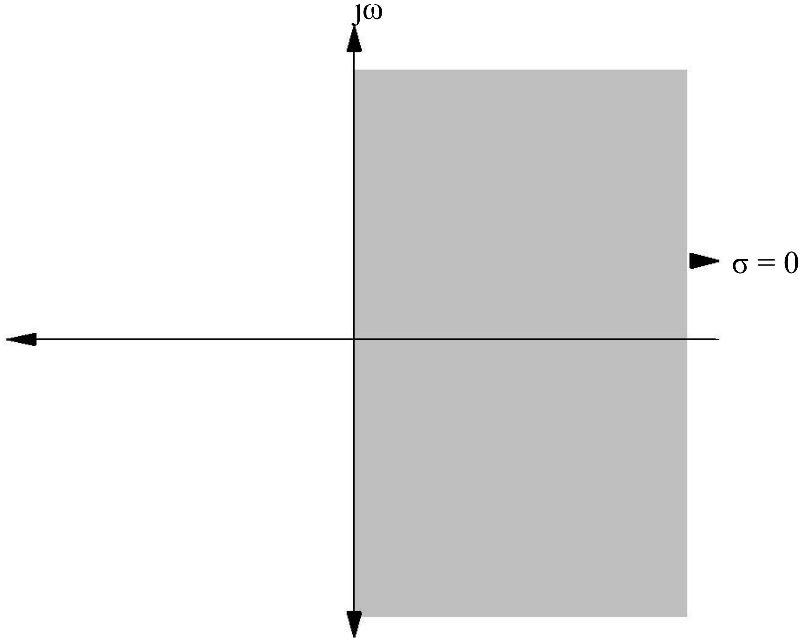

Figure 3: ROC on the complex plane for $L[e^{\alpha t}u(t)]$ for $\alpha = 0$.

3 & 4, does not contain the jω axis. However, when $\alpha < 0$, a FT for the exponential function does exist, and the jω axis is contained within the ROC of the LT. This makes sense, because the FT is a special case of the LT when $\sigma = 0$.

Specifically, if x(t) is causal and its FT exists (in the strict sense), then

$$F[x(t)] = X(s)|_{s=j\omega}$$

where $X(s) = L[x(t)]$. The existence of $F[x(t)]$ is equivalent to having the jω axis of the complex plane within the ROC of $L[x(t)]$. If jω is not within the ROC, then,

$$L[x(t)]|_{s=j\omega} =6 F[x(t)].$$

For example, lets look at the unit step,

- $F[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)$,

- L[u(t)] = $\frac{1}{s}$

Here, we see that L[u(t)] 6= F[u(t)] because of the ROC of L[u(t)] does not contain the jω axis.

And so, we see that the Laplace transform and the Fourier transform are linked together, with the Fourier transform being a special case of the Laplace transform. The Laplace transform exhibits greater explanatory power than the Fourier transform as it allows for the transformation of functions with unbounded energy.

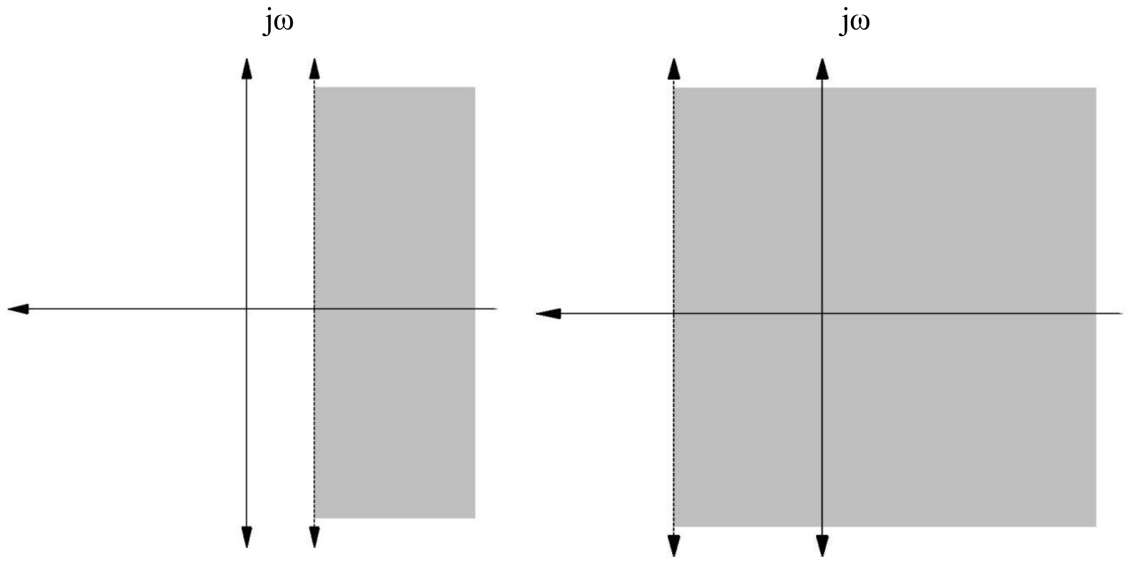

Figure 4: ROC on the complex plane for L[e$^{\alpha t}$u(t)] for α > 0.

► σ α < 0

DIFFERENCE BETWEEN FOURIER AND LAPLACE TRANSFORMATION

Laplace Transform does a real transformation on complex data but Fourier Transform does a complex transformation on real data. it means:

$$\mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt$$

$$= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-st}}{-s} f'(t)\, dt \quad \text{(by parts)}$$

$$= \left[-\frac{f(0^-)}{-s}\right] + \frac{1}{s}\mathcal{L}\left\{f'(t)\right\},$$

at the presence of some condition you can take the Fourier transformation by placing j instead of s in Laplace transform. To use Laplace to find an output given a system and input, you find the Laplace of the input X(s), and the system H(s), multiply them together to find the output Y(s), then inverse transform that to find y(t). Use Fourier Transforms you split the input signal X(s) into many pure sinusoids each of a known amplitude, phase, and frequency w, then directly find the associated output sinusoid for each of the inputs (same as the input but with a gain of |H(jw)| and with an added phase of angle(H(jw))). Fourier transformation sometimes has physical interpretation, for example for some mechanical models where we have quasi-periodic solutions (usually because of symmetry of the system) Fourier transformations gives You normal modes of oscillations. Sometimes even for nonlinear system, couplings between such oscillations are weak so nonlinearity may be approximated by power series in Fourier space. Many systems has discrete spatial symmetry (crystals) then solutions of equations has to be periodic so FT is quite natural (for example in Quantum mechanics). With any of normal modes You may tie finite energy, sometimes momentum etc. invariants of motion. So during evolution, for linear system, such modes do not couple each other, and system in one of this state leaves in it forever. Every linear physical system has its spectrum of normal modes, and if coupled with some external random source of energy (white noise), its evolution runs through such states from the lowest possible energy to the greatest.

When you Laplace transform the system, you will get the final system response, if you know the initial conditions of the system. These conditions are at t =0, can be easily obtained from the equation. The Fourier transform helps in analyzing the system response in a way different from Laplace. It breaks the signal into a number of sine and cosine waves (actually infinite), where you can have an insight to how the system is behaving by observing the amplitudes of each of the sine and cosine waves.

Fourier transforms are for converting/representing a time-varying function in the frequency domain. A laplace transform are for converting/representing a time-varying function in the "integral domain" The Laplace and Fourier transforms are *continuous* (integral) transforms of continuous functions. The Laplace transform maps a function f(t) to a function F(s) of the complex variable *s*, where

$s=\sigma+j\omega$.

Since the derivative $\dot{f}(t)$=df(t)dt maps to sF(s), the Laplace transform of a linear differential equation is an algebraic equation. Thus, the Laplace transform is useful for, among other things, solving linear differential equations.

If we set the real part of the complex variable *s* to zero, $\sigma=0$, the result is the Fourier transform F(jω) which is essentially the *frequency domain representation* of f(t).

The Z transform is essentially a discrete version of the Laplace transform and, thus, can be useful in solving *difference* equations, the discrete version of *differential* equations. The Z transform maps a sequence f[n] to a continuous function F(z) of the complex variable z=rejΩ.

If we set the magnitude of *z* to unity, r=1, the result is the Discrete Time Fourier Transform (DTFT) F(jΩ) which is essentially the frequency domain representation of f[n].

Fourier transform is defined only for absolutely integrable functions. Laplace transform is a generalisation to include all functions.

FOURIER TRANSFORM TO LAPLACE TRANSFORM

Fourier transforms map a function to a new function on the real line, whereas Laplace maps a function to a new function on the complex plane. In general, the Laplace transform is used when functions are defined on the half space t ≥0, whereas the Fourier transform is for functions defined on (-∞, ∞). Fourier as the Laplace transform can be viewed on the circle, that is |z|=1. Basically the laplace transform is used to shift the system transfer function from time domain to the frequency domain. In fourier transform we get the frequency spectrum of the signal (i.e) the various frequencies in the given composite signal and their relative amplitude.

Laplace Transform does a real transformation on complex data but Fourier Transform does a complex transformation on real data. it means:

at the presence of some condition you can take the Fourier transformation by placing j instead of s in Laplace transform. To use Laplace to find an output given a system and input, you find the Laplace

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\,dt$$

$$= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-st}}{-s} f'(t)\,dt \quad \text{(by parts)}$$

$$= \left[-\frac{f(0^-)}{-s}\right] + \frac{1}{s}\mathcal{L}\{f'(t)\},$$

of the input X(s), and the system H(s), multiply them together to find the output Y(s), then inverse transform that to find y(t). Use Fourier Transforms you split the input signal X(s) into many pure sinusoids each of a known amplitude, phase, and frequency w, then directly find the associated output sinusoid for each of the inputs (same as the input but with a gain of |H(jw)| and with an added phase of angle(H(jw))). Fourier transformation sometimes has physical interpretation, for example for some mechanical models where we have quasi-periodic solutions (usually because of symmetry of the system) Fourier transformations gives You normal modes of oscillations. Sometimes even for nonlinear system, couplings between such oscillations are weak so nonlinearity may be approximated by power series in Fourier space. Many systems has discrete spatial symmetry (crystals) then solutions of equations has to be periodic so FT is quite natural (for example in Quantum mechanics). With any of normal modes You may tie finite energy, sometimes momentum etc. invariants of motion. So during evolution, for linear system, such modes do not couple each other, and system in one of this state leaves in it forever. Every linear physical system has its spectrum of normal modes, and if coupled with some external random source of energy (white noise), its evolution runs through such states from the lowest possible energy to the greatest.

CONCLUSION

One of the first questions we ask ourselves when talking about the Fourier transform and the Laplace transform is “Why are we taking these transforms, anyway?”. Essentially, why do we want to map one function to another and to what end? The answer is “It Depends.” And it really does. Different transforms give us different ways of viewing a particular function or signal (which, in the CT case, is just a function defined over time.) You can think of a transform as a microscope, or perhaps even a telescope (depending on how big you imagine your function to be) which will show you more about the inner workings of a function which might otherwise go unobserved.

The features that a transform highlights depends on the construction of the transform itself. Micro-scopes and microphones tell us two different kinds of information. While we look at just a few transforms in this course, there exists an entire menagerie of transforms which are used all the time to analyze functions and signals: cosine transforms, wavelets, curve lets, noise lets, short-time FT's, chaplets, to name just a few.

In Signals & Systems, we concern ourselves mostly with the Fourier and Laplace transforms, including their discrete versions (for practical implementation). These two transforms, while similar, have some distinct differences that are worth mentioning, hence this document. Fourier transforms map a function to a new function on the real line, whereas Laplace maps a function to a new function on the complex plane. In general, the Laplace transform is used when functions are defined on the half space t ≥0, whereas the Fourier transform is for functions defined on (-∞, ∞). Fourier as the Laplace transform can be viewed on the circle, that is |z|=1. Basically the laplace transform is used to shift the system transfer function from time domain to the frequency domain. In fourier transform we get the frequency spectrum of the signal (i.e) the various frequencies in the given composite signal and their relative amplitude.Laplace Transform does a real transformation on complex data but Fourier Transform does a complex transformation on real data. it means:

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\,dt$$

$$= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-st}}{-s} f'(t)\,dt \quad \text{(by parts)}$$

$$= \left[-\frac{f(0^-)}{-s}\right] + \frac{1}{s}\mathcal{L}\{f'(t)\},$$

at the presence of some condition you can take the Fourier transformation by placing j instead of s in Laplace transform. To use Laplace to find an output given a system and input, you find the Laplace of the input X(s), and the system H(s), multiply them together to find the output Y(s), then inverse transform that to find y(t). Use Fourier Transforms you split the input signal X(s) into many pure sinusoids each of a known amplitude, phase, and frequency w, then directly find the associated output sinusoid for each of the inputs (same as the input but with a gain of |H(jw)| and with an added phase of angle(H(jw))). Fourier transformation sometimes has physical interpretation, for example for some mechanical models where we have quasi-periodic solutions (usually because of symmetry of the system) Fourier transformations gives You normal modes of oscillations. Sometimes even for nonlinear system, couplings between such oscillations are weak so nonlinearity may be approximated by power series in Fourier space.

Many systems has discrete spatial symmetry (crystals) then solutions of equations has to be periodic so FT is quite natural (for example in Quantum mechanics). With any of normal modes You may tie finite energy, sometimes momentum etc. invariants of motion. So during evolution, for linear system, such modes do not couple each other, and system in one of this state leaves in it forever. Every linear physical system has its spectrum of normal modes, and if coupled with some external random source of energy (white noise), its evolution runs through such states from the lowest possible energy to the greatest.

REFERENCE:

[1]. Shin, C., & Cha, Y. H. (2009). Waveform inversion in the Laplace—Fourier domain. Geophysical Journal International, 177(3), 1067-1079.

[2]. Durbin, F. (1974). Numerical inversion of Laplace transforms: an efficient improvement to Dubner and Abate's method. The Computer Journal, 17(4), 371-376.

[3]. Beerends, R. J. (2003). Fourier and Laplace transforms. Cambridge University Press.

[4]. Bracewell, R. N., & Bracewell, R. N. (1986). The Fourier transform and its applications (Vol. 31999). New York: McGraw-Hill.

[5]. Bayın, Ş. S. Fourier and Laplace Transforms. Essentials of Mathematical Methods in Science and Engineering, 607-636.

[6]. De Concini, C., & Procesi, C. (2011). Fourier and Laplace Transforms. In Topics in Hyperplane Arrangements, Polytopes and Box-Splines (pp. 69-75). Springer New York.

[7]. Crump, K. S. (1976). Numerical inversion of Laplace transforms using a Fourier series approximation. Journal of the ACM (JACM), 23(1), 89-96.

[8]. Chill, R. (1998). Tauberian theorems for vector-valued Fourier and Laplace transforms. Studia Mathematica, 128(1), 55-69.

[9]. Arendt, W., Batty, C. J., Hieber, M., & Neubrander, F. (2011). Vector-valued Laplace transforms and Cauchy problems (Vol. 96). Springer Science & Business Media.

[10]. Akbarzadeh, A. H., Abbasi, M., & Eslami, M. R. (2011). Dynamic analysis of functionally graded plates using the hybrid Fourier-Laplace transform under thermomechanical loading. Meccanica, 46(6), 1373-1392.