International Journal of Research
Available at https://edupediapublications.org/journals

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 05
April 2017

Numerical Solution of Differential Equations

Ahmed Jamal Skakir Almansory

Applied mathematics

Abstract:

To study the techniques for solving differential equations based on numerical approximations were developed before programmable computers existed. Differential equations are an important part of many areas of mathematics, from fluid dynamics to celestial mechanics. Many mathematicians have studied the nature of these equations for hundreds of years and there are many well-developed solution techniques. Except for a few special cases, differential equations cannot normally be solved analytically. Instead, there are many numerical methods which have been developed to provide solutions.

Keywords: Differential Equations, Fluid Dynamics, Numerical Method.

INTRODUCTION:

Differential equations to physical problems in the analysis of motion. Here, there are two common differential equations. The first relates the displacement along a one-dimensional path, s, to the velocity, v; the second, which is Newton's second law relates the displacement to the applied force divided by the mass, F/m. The symbol t indicates the time.

$$\frac{ds}{dt} = v \qquad\qquad \frac{d^2 s}{dt^2} = \frac{F}{m}$$

Another application of differential equations is in electrical circuits. The current, I, in a circuit with a capacitance, C, an inductance, L, a resistance, R, and an applied voltage, V(t) is governed by the following differential equation. (In this equation, V(t) is known.)

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C} I = \frac{dV(t)}{dt}$$

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 05
April 2017

Differential equations are classified in terms of the highest order of the derivative that appears in the equation. Thus, equation [2] is a second order differential equation. The two differential equations in [1] are, respectively, first-order equation and second-order differential equations. The equations in [1] and [2] are linear differential equations. In these equations the dependent variable, and all its derivatives, appear to the first power only.

We can use the definition of velocity to write Newton's second law as two first-order differential equations.

$$\frac{ds}{dt} = v \qquad\qquad \frac{dv}{dt} = \frac{F}{m}$$

Similarly, we can use the definition of voltage drop across the inductor, e_L as the Inductance times the first derivative of the current, to rewrite equation [2] as the following pair of equations.

$$e_L = L\frac{dI}{dt} \qquad\qquad \frac{de_L}{dt} + \frac{e_L}{R} + \frac{1}{C}I = \frac{dV(t)}{dt}$$

In this system of equations we have one independent variable, t, and two dependent variables, I and e_L. This approach of writing second-order equations as sets of first-order equations is possible for any higher order differential equation. We will use it subsequently to apply algorithms designed for the analysis of first order equations to systems of higher order equations.

Some differential equations can be solved by simple integration. An example of this is shown below.

$$\frac{ds}{dt} = v(t) \qquad so\ that \quad \int ds = \int v(t)\,dt \quad or \quad s = \int v(t)\,dt + C$$

International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 05
April 2017

The constant of integration, C, can be found if one point in the relationship, typically called an initial condition, ($s_{initial}$, $t_{initial}$) is known. With the initial condition, we can find the value of s for any value of t by the following integration.

$$\int_{s_{initial}}^{s} ds = s - s_{initial} = \int_{t_{initial}}^{t} v(t')\,dt' \qquad or \qquad s(t) = s_{initial} + \int_{t_{initial}}^{t} v(t')\,dt'$$

We have used the dummy variable, t', in the integral to indicate that the final dependence of s(t) depends on the upper limit of the integral.

We could perform the integration in equations because the derivative expression was a function of the time only. We are interested in the more general problem of what happens when the derivative in equation [4] is a function of both time and displacement. That is we are interested in solving the general problem

$$\frac{ds}{dt} = v(t, s)$$

We can try to write this as we did in equation [4] or [5], but we cannot perform the integration because v is a function of both s and t>

$$\int ds = \int v(t, s)\,dt \quad or \quad s = s_{initial} + \int_{t_{initial}}^{t} v(t', s)\,dt'$$

There are may cases in which we can solve differential equations like equation analytically. However, when we cannot do so, we have to find numerical methods for solving this equation.

Solving Differential Equations

Solving differential equations is an art and a science. There are so many different varieties of differential equations that there is no one sure-fire method that can solve all differential equations exactly (i.e. coming up with a closed form for a solution

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 05
April 2017

function, such as $y = 3x + 5$). . There are, however, a number of numerical techniques that can give approximate solutions to any desired degree of accuracy. Using such a technique is sometimes necessary to answer a specific question, but often it is knowledge of an exact solution that leads to better understanding of the situation being described by the differential equation. In our math 21a classes, we will concentrate on solving ODEs exactly, and will not consider such numerical techniques. However, if you are interested in seeing some numeric techniques in action then you might consider trying solving some differential equations using the Mathematica program.

Note that to solve example (1) $y' = 3$, we could have simply integrated both sides. This follows from the very basic idea that anytime we are given two things that are equal, then as long as we do the same thing to one side of an equation as to the other then equality still holds. For an obvious example of this principle in action, if $x = 2$, then $x + 4 = 2 + 4$ and $3x = 6$. So solving the differential equation $y' = 3$, is pretty straightforward, we just have to integrate both sides: $\int y' dx = \int 3 dx$. The fundamental theorem of calculus then tells us that the integral of the derivative of a function is just the function itself up to a constant, i.e. that $\int y' dx = y + c_1$, and also that $\int 3 dx = 3x + c_2$ where we represent different constants by writing c_1 and c_2, to distinguish them from each other. Yes, all these subscripted constants can appear odd, but there will be times when keeping good records of new constants that come along while we're solving differential equations will be especially important. Think of it as keeping track of "+" or "–" in an equation - yes it can be somewhat annoying at times, but clearly it's critical!

International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 05
April 2017

Finally, then, we have $y = 3x + (c_2 - c_1)$, which we simplify as $y = 3x + c$, since until an initial condition is given, we don't actually know the value of any of these constants, so we might as well lump them together under one name (Hadamard, J. (2014)).

Separation of Variables Technique

Can we follow the same approach for the other differential equations? Well, usually no, we can't. The reason this worked so nicely in our first example was that the two sides of the equation were neatly separated for us to do each of the integrations. Suppose we tried the same thing for the differential equation in (4), i.e. $P'(t) = k\,P(t)$. Here we could try to integrate both sides directly, so that we write

$$\int P'(t)\,dt = \int kP(t)\,dt$$

Although we can easily do the left side integral, and just end up with $P(t) + c$, integrating the right-hand side leads us in circles, how can we integrate a function such as $P(t)$ if we don't know what the function is (finding the function was the whole point, after all!). It appears that we need to take a different approach. Since we don't know what $P(t)$ is, let's try isolating everything involving $P(t)$ and its derivatives on one side of the equation:

$$\frac{1}{P(t)} P'(t) = k$$

Now let's try to integrate both sides with respect to t again:

$$\int \frac{1}{P(t)} P'(t)\,dt = \int k\,dt$$

The reason we're in better shape now is that the right-hand side integral is trivial, and the left-hand side integral can be taken care almost directly, with a quick substitution: if $u = P(t)$ then $du = P'(t)\,dt$ and so (7) becomes

$$\int \frac{1}{u}\,du = \int k\,dt$$

so that, $\ln|u| = \ln|P(t)| = kt + c_1$

where we've gone ahead and combined constants on one side. Now solving for the function *P(t)*, we find that

$$P(t) = \pm e^{kt + c_1} = \pm e^{c_1} e^{kt} = Ce^{kt}$$

where we've simply lumped all the unknown constants together as *C*. Does this really satisfy the original differential equation (4)? Check and see that it does. If someone had added an initial condition such as $P(0) = 1000$, then this extra information would allow us to solve and find out the constant $C = 1000$ (remember that the other constant, *k*, would necessarily be known before we started as it shows up in the original differential equation).

This technique of splitting up the differential equation by writing everything involving the function and its derivatives on one side, everything else on the other, and then finally integrating is called ***separation of variables***. It's a really useful trick! One notational shortcut you can use as you go through the separation of variables technique is to write $P'(t)$ as $\frac{dP}{dt}$ in the original differential equation (4) which then becomes

$$\frac{dP}{dt} = kP$$

Next, when you separate the variables, treat the $\frac{dP}{dt}$ as if it were a fraction (you've probably seen this type of thing done before – remember, this is only a notational shortcut, the derivative $\frac{dP}{dt}$ is one whole unit, of course, and not a fraction!). Thus to separate the variables we get

$$\frac{1}{P}dP = kdt$$

Now the integration step looks as if it is happening with respect to P on the left hand side and with respect to x on the right hand side:

$$\int \frac{1}{P}dP = \int kdt$$

and after you do these integrations, you're right back to equation (9), above.

When you can use the separation of variables technique, life is good! Unfortunately, as with the integration tricks you learned in single variable calculus, it's not the case that one trick will take care of every possible situation. In any case, it's still a really good trick to have up your sleeve!

Examples

The following question cannot be solved using the algebraic techniques we learned earlier in this chapter, so the only way to solve it is numerically.

Solve using Euler's Method:

dx/dy=sin(x+y)−e^x

Y(0) = 4

Use h = 0.1

Solution:

We start at the initial value (0,4) and calculate the value of the derivative at this point. We have:

$$\frac{dy}{dx} = \sin(x + y) - e^x,$$

$$= \sin(0 + y) - e^0, = -1.75680249531$$

We substitute our starting point and the derivative we just found to obtain the next point along.

$$y(x + h) \approx y(x) + hf(x, y)$$

$$y(0.1) \approx 4 + 0.1(-1.75680249531)$$

$$\approx 3.82431975047$$

Example:

The numerical methods for solving ordinary differential equations are methods of *integrating a system of first order differential equations*, since higher order ordinary differential equations can be reduced to a set of first order ODE's. For example,

$$p(x)\frac{d^2y}{dx^2}+q(x)\frac{dy}{dx}=r(x)$$

$$\text{Let } y(x)=y_1(x) \text{ and } \frac{dy_1}{dx}=y_2(x)$$

$$\Rightarrow \begin{cases} \dfrac{dy_1}{dx}=y_2(x) \\ \dfrac{dy_2}{dx}=\left[r(x)-q(x)y_2(x)\right]/p(x) \end{cases}$$

An nth order ordinary differential can be similarly reduced to

$$\frac{dy_k(x)}{dx}=f_k(x,y_1,y_2,\ldots,y_n) \text{ where } k=1,2,\ldots,n$$

CONCLUSION:

In this study we conclude we start by discussing what differential equations are. Our discussion emphasises the simplest ones, the so-called first order equations, which only involve the unknown function and its first derivative. We then consider how first order equations can be solved numerically by the simplest method, namely Euler's method. We analyse the error in Euler's method, and then introduce some more advanced methods with better accuracy. After this we show that the methods for handling one equation in one unknown generalise nicely to systems of several equations in several unknowns.

Reference:

[1]. L. Jay. "Convergence of Runge-Kutta methods for differential-algebraic systems of index3", *Applied Numerical Mathematics* 17 (1995), pp. 97–118.

[2]. H. Keller. *Numerical Solution of Two-Point Boundary Value Problems*, Regional Conf.Series in Appl. Maths. **24**, SIAM Pub., Philadelphia, PA, 1976.

[3]. H. Keller and S. Antman, eds. *Bifurcation Theory and Nonlinear Eigenvalue Problems*, Benjamin, New York, 1969.

[4]. C.T. Kelley. *Solving Nonlinear Equations with Newton's Method*, SIAM Pub., Philadelphia, 2003.

[5]. W. Kelley and A. Peterson. *Difference Equations*, 2nd ed., Academic Press, Burlington, Massachusetts, 2001.

[6]. R. Kress. *Numerical Analysis*, Springer-Verlag, New York, 1998.

[7]. J. Lambert. *Computational Methods in Ordinary Differential Equations*, JohnWiley, New York, 1973.

[8]. P. L¨otstedt and L. Petzold. "Numerical solution of nonlinear differential equations with algebraic constraints. I.Convergence results for backward differentiation formulas", *Mathematics of Computation* **46** (1986), pp. 491–516.

[9]. J.Marsden and T. Ratiu. *Introduction to Mechanics and Symmetry*, Springer-Verlag, New York, 1999.

[10]. R. M¨arz. "Numerical methods for differential algebraic equations", *Acta Numerica 1992*, Cambridge University Press, 1992.

[11]. D. Melgaard and R. Sincovec. "Algorithm 565: PDETWO/PSETM/GEARB: Solution of systems of two-dimensional nonlinear partial differential equations", *ACM Trans. Math.Software* **7** (1981), pp. 126–135.

[12]. R. Miller. *Nonlinear Volterra Integral Equations*, Benjamin Pub., 1971.

[13]. L.R. Petzold. "A description of DASSL: A differential-algebraic system solver", in R. S.

[14]. Stepleman, editor, *Scientific Computing*, pp. 65–68. North-Holland, Amsterdam, 1983.

[15]. L. Petzold, L. Jay, and J. Yen. "Numerical solution of highly oscillatory ordinary differential equations", *Acta Numerica* **6** (1997), Cambridge University Press.

[16]. E. Platen. "An introduction to numerical methods for stochastic differential equations", *Acta Numerica* **8** (1999), Cambridge University Press.

[17]. Quarteroni, R. Sacco, and F. Saleri. *Numerical Mathematics*, Springer-Verlag, New York, 2000.

[18]. H.Keller.*NumericalMethods for Two-Point Boundary Value Problems,*Dover, New York, 1992 (corrected reprint of the 1968 edition, Blaisdell, Waltham, MA).

[19]. L.B. Rall and G.F. Corliss. “An introduction to automatic differentiation”, in *Computational Differentiation* (Santa Fe, NM, 1996), pp. 1–18. SIAM, Philadelphia, PA, 1996.