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A Survey on Generalized Inner Product Spaces and Strong Topologies

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Abstract

In This project we have discussed the concept of orthognality in normed space, which is an extension of the concept orthognality in Hilbert spaces. This concept has been used by Giles [2] to obtain some interesting result in semi - inner product spaces .We have presented some results on Hilbert algebras due to Ambrose [1], Keown [3], and Ingelstam[4] and [5]many of these results have been extend to semi - 1nner product algebras by Hussein and Malviya [6]and[7].we have carried out a study of generalized inner product and semi-inner product spaces and algebras.

Some New Class of Hilbert Algebra

Ambrose introduced and studied what he called, H∗- algebras. The consideration of these H∗ - algebras arose from a consideration of the L_2 - algebras of a compact group. Known introduced a new class of commutative Hilbert algebras which is in a sense a generalization of the class of H∗- algebras .this essential difference between the works of AMBROSE AND Keown is that the latter does not obtain the decomposition of the algebras in to orthogonal subspaces each of which is a minimal left ideal . This chapter presents the result of Ambrose ,Keown and ingelstam . many of these results will be extended to semi - inner product algebras.

1. H∗- ALGEBRA

Definition 1: A banach algebra A whose underlying banach space is a Hilbert space is called an H∗- algebra if for each x in A, there is an element x ∗ in A, called adjoint of x, such that (xy,z) = (y, x∗z) and (yx, z) = (y,zx∗) for all y and Z in A.

Remark 1: We will see later that x∗ need not be unique. The proof of following result is simple and hence omotted.

Proposition 1. Let A be an H∗-algebra and x∗ adjoint of x in A. Then

(a) x∗∗ = x.

(b) (xy)∗ = y∗x∗, x,y in A.

(c) (ax+by)∗= ax∗+by∗, x,y in A and a,b in C

(d) xx∗(or x∗x) is self - adjoint. Here are some examples of H∗-algebras

EXAMPLE 1: Let J be an arbitrary set of elements, and define

B = { a(i,j) ∊ C, ∑i j | a (i, j)| < ∞, (i, j) in JxJ}.

With (a+b) (i,j) = a (i,j) + b (i,j)

(a,b) (i,j) = ∑a(i,k)b(k,j),

(λa)(i,j), λ in C

(a,b) = a ∑i j a (i, j) b(i, j), L ≥ 0 and a∗ (i,j) = a⁻ (j,i) B becomes an H∗ - algebra. If n is the cardinal number of J then B is called a ful matrix H∗-algebra of order in the subalgebra of all diagonal elements of a full matrix algebra is also another example of an H∗- algebra.

EXAMPLE 2: Let G be a compact topological group. Consider the space L_2 (G) of complex - valued functions of Integral Square with respect to the Haar measure of G. Define in L_2 (G),

(f + g) (x) = f(x) + f(x),

(fg)(x) =∫ f (xy^{-1}) g(y)dy

(λf)(x) = λf(x)

(f,g) =∫ f(x) g ⁻ (y) dy, and (f∗)(x) = f⁻ (x^{-1}). Then L_2 (G) becomes an H∗-algebra.

Generalized Inner Product Spaces

In this chapter we discuss a straight forward algebraic generalization of inner product spaces, called generalized inner product (in short, g.i.p.) spaces, and we study various topologies in these spaces in such a manner that locally convex spaces are obtained .

All the results of this chapter are due to prugovecki .

1 G.I.P Spaces

Definition 1. A vector space E is called a generalized inner product (abbreviated to g.i.p.) space if

(a) There is subspace M of E which is a inner product space and ,

(b) There is a set $\mathcal{L}$ of linear operators on E such that

(i) $\mathcal{L}E\subset M$ and

(ii) Tx=0 , for all T in $\mathcal{L}$ implies x=0

We denote such a g.i.p. space by the triple (E,$\mathcal{L}$M,)

Clearly an inner product space is a g.i.p . Space. The following example shows that g.i.p. spaces from a proper generalization of inner product spaces.

Example 1. Let E=Co(R) be the family of all real continuous functions on R .let M consists of all squares integrable in E. We define the inner product in M as follows

$$(x,y)=\int x(t)\ y(t)dt$$

We take $\mathcal{L}$ to be the family of all projections p(I) , defined by

$$P(I)\ (x))(t)= \chi_1\ (t)x(t)$$

Where I is the finite non degenerate interval and x_1 (t) is the characteristic function of I . (E, $\mathcal{L}$, M) is a g.i.p space which is not an inner product space

Proposition 1 .let (E,$\mathcal{L}$' M) be a g.i.p space and x in E . if (Tx,y) =0 for all y in M and T in $\mathcal{L}$, then x=0

Proof .clearlyTx is in M . hence if (Tx,y) =0 for all y in M , then Tx =0 for all T in $\mathcal{L}$. but then x=0 by the definition 1 .

Corollary 1. Let (E,$\mathcal{L}$,M) be a g.i.p .space an x in E . (Tx,Tx) =0 for all T in $\mathcal{L}$ the x=0

Proof .if (Tx,Tx)=0 then Tx=0 . now proposition 1 applies to the result

2 . Strong Topologies.

Definition 2 .let (E,$\mathcal{L}$,M) be a g.i.p space for each x in E , the sets of the form

$V(x;T_1,.......T_n;\varepsilon)=\{y\varepsilon E;(T_k(y-x),T_b(y-x))^2 <\varepsilon$, $k=1,2,....n\}$ for all $\varepsilon<o$, $T_1,..........T_n$ in $\mathcal{L}$, n=1,2,......, from a neighborhood basis for a topology in E called the strong topology

Lemma 1. Each $V(0;T_1,.......T_n;\varepsilon)$ is circled and convex .

Proof . Let $x\in V(0;T_1,.......T_n;\varepsilon)$

Then

$(T_kx,T_kx)^{1/2}<\varepsilon$, $k=1,......,n$.

Hence for all $|\lambda|\leq$,

$(T_k(\lambda x),T_k(\lambda x)^{1/2} = \|T_k(\lambda x)\| =|\lambda|.\|T_k(x)\|$

$= |\lambda|(T_kx,T_kx)^{1/2}<\varepsilon$, $k=1,......,n$.

This implies that

$\lambda x\in V(0;T_1,.......T_n;\varepsilon)$

And $V(0;T_1,.......T_n;\varepsilon)$ is circled. To show that it is convex, let

$x_1,x_2\in V(0;T_1,.......T_n;\varepsilon)$

and $0\leq\lambda\leq1$.

Then

$(T_k(\lambda x_1+(1-\lambda)x_2), T_k(\lambda x_1)+(1-\lambda)x_2))^{1/2}$

$=\|T_k(\lambda x_1+(1-\lambda)x_2)\|$

$=\|\lambda T_kx_1+(1-\lambda)T_k x_2)\|$

$=\lambda|(T_kx_1,T_k x_1)^{1/2} +(1-\lambda)(T_kx_2,T_k x_2)^{1/2}<\varepsilon$

This implies that

$\lambda x_1+(1-\lambda)x_2\varepsilon V(0;T_1,.......T_n;\varepsilon)$

And so $V(0;T_1,.......T_n;\varepsilon)$ is convex.

Lemma 2 . (E,$\mathcal{L}$,M) be g .i.p .space .If u is a topology on E for which the sets V (x; T ;ε) are neighbourhoods of x for all $\varepsilon>o$, T in $\mathcal{L}$, then u is Hausdorff.

Proof .suppose u is not Haussdorff . then there exists atleast to distinct elements x_1

and x_2 in E for which any two neighbourhoods have common points. thus for any $V(x_1;T;1/n)$ and $V(x_2;T;1/n)$ there is atleast one y_n in E such that

$y_n \in V(x_1;T;1/n) \cap V(x_2;T;1/n)$.

Hence

$$(T(x_1-y_n),T(x_1-y_n))^{1/2}<1/n$$

$$(T(x_2-y_n),T(x_2-y_n))^{1/2}<1/n$$

Then

$$(T(x_1-x_2),T(x_1-x_2))^{1/2}=\| T(x_1-x_2)\|$$

$\leq 2/n$

From this it follows that

$$T(x_1-x_2)=0$$

For all T in $\mathcal{L}$, and $x_1=x_2$ which is contradiction.

Theorem 1 . Let $(E,\mathcal{L},M)$ be a g.i.p .space . E , equipped with the strong topology , is a Hausdorff locally convex space

Proof .we first show that the strong topology is compatable with the vector operations . for any $V(x_1+x_2;T_1,.......T_n;\varepsilon)$ we show that

$$V(x_1;T_1,.......T_n;\varepsilon/2)+V(x_2;T_1,.......T_n;\varepsilon/2)$$

$\subset V(x_1+x_2;T_1,.......T_n;\varepsilon)$.

Let

$$Y_1 \in V(x_1;T_1,.......T_n;\varepsilon/2)$$

And

$$Y_2 \in V(x_2;T_1,.......T_n;\varepsilon/2)$$

Then

$$(T_k(y_1+y_2-x_1-x_2),T_k(y_1+y_2-x_1-x_2))^{1/2}$$

$\|T_k(y_1+y_2-x_1-x_2)\|$

$\leq \|T_k(y_1-x_1)\|+\|T_k(y_2-x_2)\|<\varepsilon/2+\varepsilon/2=\varepsilon$,

Where $k=1,....,n$. thus the operations of the vector summation is continuous that similarly we can show that the operation of multiplication, by a scalar is continuous. The rest follows from lemma 1 and 2

Definition 3 .let $(E,\mathcal{L},M)$ be a g.i.p. space . the family of sets

$$V(x;T_1,....;\varepsilon)=\cap V(x,T_k;\varepsilon)$$

Where the T_k are in $\mathcal{L}$,and $\varepsilon>0$, constitute a neighbourhood basis for a topology on E which we call the ultra-strong topology .

Clearly the ultra-strong topology is finer than the strong topology .

Proposition 2 . A g.i.p.space $(E,\mathcal{L},M)$ with strong (ultra-strong) topology is metrizable if there is a countable subset $\mathcal{B}$ of $\mathcal{L}$ which has the property that for any T in $\mathcal{L}$ there is an S in L , L being the linear manifold generated by $\mathcal{B}$ such that

$$(Tx,Tx)^{1/2}\leq(Sx,Sx)^{1/2}$$

For all x in E

Proof .it is sufficient to show that the family of sets

$V(0;S_1,.......S_n;1/n)$; $S_1,.......S_n\in\mathcal{B}$, k,n=1,2,.....

Is a neighbourhood basis at o for the strong topology for every T in $\mathcal{L}$ we can find an S in L for which

$$V(0;S;\varepsilon)\subset V(0;T;\varepsilon)$$

Clearly we have

$$S=\lambda_1 S_1 +..........+\lambda_k S_{k,}\ ); \ S_1,.......S_n\in\mathcal{B}$$

And so

$$(Sx,Sx)^{1/2} =\|Sx\|\leq|\lambda|.\|S_1x\|+........+|\lambda_k|.\|S_kx\|$$

For all x in E.Thus ,If we choose an integer n such that

$$1/n\leq\varepsilon/k|\lambda_1|,........,1/n\leq\varepsilon/k|\lambda_k|$$

We have

$$V(0;S;\varepsilon)\supset V(0;S_1;1/n)\cap.......\cap V(0;S_k;1/n)$$

$$= V(0;S_1,.......S_n;1/n)$$

The proof for ultra-strong topology can similarly be obtained.

Corollary 2. The g .i.p .space C_o(Example 1) is metrizable in the strong and ultra-strong topologies.

Proof. Choose the countable family

$$\mathcal{B}=\{P([n,n+1]);n=0,\pm1,\pm2,.....\}$$

Of projectors from $\mathcal{L}$. If P(I) is in $\mathcal{L}$, then I is a finite interval and consequently we can defined integers m_1 and m_2 , $m_1<m_2$ such that

$$I \subset \ U\ [n,n+1).$$

Hence , we have, for any x in C_o ,

$(P(I)x, P(I)x)^{1/2} \leq (Sx,Sx)^{1/2}$,

where$S=\Sigma P([n,n+1)) \in$ L.

3 Dual Spaces and Weak Topologies.

Let $(E,\mathcal{L},M)$ be a g.i.p. space . we can assign to each T in $\mathcal{L}$ and each $\xi \in M$, a linear functional

$\phi(x; T,\xi)= (Tx,\xi)$

On E . Denote by F_o the family of all such functional . $F_o$is not a vector space , in general . let F be the vector space (over the same field as that of E) spanned by F_o.

Proposition 3 . F and E constitute a dual pair .

Proof .if$\phi(x)$ =0 for all ϕ in F .then

$(Tx,\xi)=0$

For allξ in M and T in $\mathcal{L}$ .but then , ,by proposition I,x=0 , conversely , if for a given ϕ_o in F. $\phi_0(x)=0$ for all x in E then $\phi_0=0$

Notation . We write

$<x,\phi>=\phi(x)$. $\phi \in F, x \in E$.

Clearly $<x,\phi>$ is a bilinear functional on F and E .

Proposition 4 : each ϕ in F is continuous in the strong (ultra-strong) topology .

Proof .let$\varepsilon>0$. Now $|\phi(x;T,\xi) -\phi(x_0 ;T,\xi)|= |T(x-x_0),\xi)|$

$\leq \|T(x-x_0)\|.\|\xi\|<\varepsilon$.

Whenever $(T(x-x_0), T(x-x_0)^{1/2}=\|T(x-x_0)\|<\varepsilon/\|\xi\|)$.

i.e., for all

$x \in V(x_0;T, \varepsilon/\|\xi\|)$.

Thus each ϕ in F_0 is continuous functional on E in the strong (ultra - strong) topology.

Hence the continuity of each ϕ in E follows .

Remark 1. F is contained in the vector space conjugate to E with the strong topology .

Theorem 2. Let $(E,\mathcal{L},M)$ be a g.i.p. space . If M is finitedimensional ,then F is

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isomorphic to the vector space conjugate to E with the strong topology .

Proof . All that we need to show is that if f(x) is a continuous linear functional on E with the strong topology , then f is in F . For a given $\varepsilon, 0<\varepsilon<1$. We can find a neighbourhood

$V(0;T_1,.......T_k;\delta)$ at 0 for the strong topology such that

$|f(x)|<\varepsilon$

For all x. On the other hand , since M is finite - dimensional, there is a basis ξ_1 ,...... ξ_n in m which spans M_o consider the finite set of continuous linear functional

$$(*) \quad \phi_1(x) = \phi(T_1x,\xi_1) \ , \ i=1,....,n \ , \ j=1,....,k.$$

If f(x) were independent of the functionals (∗). Then there shoud be an x_1 in E for which $f(x_1)=1$, $\phi_1(x_1)=.....=\phi_n k(x_1)=0$

but then $T_1x_1=.......=T_kx_1=0$

Because $\xi_{1....}\xi_n$ is a basis in M, and so $X_1\in V(0;T_1,.......T_k;\delta)$.

This implies that $|f(x_1)|< 1$

Which is contradiction .this completes the proof .

The following example demonstrates the existence of a g.i.p space $(E,\mathcal{L},M)$ with M finitr -dimensional.

Example 2. Let Me be the family of all one -row infinite matrices with real elements $(a_1,a_2,.........)$,and M the one dimensional space of all one - row real matrices $(a_1,0,0,......)$ in which only the first element isnon -vanishing . we adopt the customary inner product in M . we choose

$\mathcal{L}=(P_1,P_2,....)$

Where P_n is the operator

$P_n(a_1,a_2,......,a_n)=(a_n,o,...,0..)$.

Then $(e,\mathcal{L},M)$ is a g.i.p.space with M one dimensional space .

As usual, the weak topology on E is the coarsest topology in which all the linear functional from F are continuous .As is well-known. The family of all subsets $W(x_0;\phi_1,.....\phi_n)$ of E where $.W(x_0;\phi_1,.....\phi_n)=\{x\in E;|\phi_k(x-x_0)|<1,k=1,.......n\}$

Corresponding to all $\phi_1,\ldots..\phi_n$ in F ,n=1,2,....... is a neighbourhood basis at x_0. Since F_0 generates F , the family of all neighbourhoods

$W(o;\xi_1 T_1\ldots..\xi_n,T_n)=\{x\in E;|(T_k x,\xi_k)|<1,k=1,\ldots.n\}$

$= \{x\in E;|(T_k x,\xi_k)|<1,k=1,\ldots.n\}$

Corresponding to all $\xi_{1\ldots.}\xi_n$ in M, $T_1\ldots..,T_n$ in $\mathcal{L}$,n=1,2,....., is also a neighbourhood basis at 0. Since E and F constitute a dual pair ,Eis a Hausdorff topological space in the weak topology .

In view of the general properties of weak topologies we have the following

Proposition 5. The space E equipped with the weak topology is a Hausdorff locally convex space .

Definition 4 .the family of all sets

$W(x_0;\phi_1,\ldots..\phi_n)$

$=\{x\in E;|\phi_k(x-x_0)|<1,k=1,\ldots....\}$

Corresponding to all sequence $,\phi_1,\ldots..\phi_{n,}$ .... in F from a neighbourhood basis x_0 for a topology on A which we call the infra -weak topology .

Clearly the infra- weak topology is finier than the weak topology and hence Hausdorff

Proposition 6. The space E equipped with the infra- weak topology is a Hausdorff locally convex space .

Proof . It is easy to check that the infra-weak topology is compatable with the vector operations on E .furthermore it can easily be shown that the sets $W(x_0;\phi_1,\ldots..\phi_n)$ are convex .

Hence the result follows.

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