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A Study on Analytic Function of Complex Analysis

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ABSTRACT:

To study and analyse the analytic functions in complex analysis system. A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function". Many mathematicians prefer the term "holomorphic function" (or "holomorphic map") to "analytic function", while "analytic" appears to be in widespread use among physicists, engineers, and in some older texts. If a complex function is analytic on a region R, it is infinitely differentiable in R. A complex function may fail to be analytic at one or more points through the presence of singularities, or along lines or line segments through the presence of branch cuts. A complex function that is analytic at all finite points of the complex plane is said to be entire. A single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities goes to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities), is called a homomorphic function. The study also analysis different approaches to the concept of analyticity. One definition, which was originally proposed by Cauchy, and was considerably advanced by Riemann, is based on a structural property of the function the existence of a derivative with respect to the complex variable, i.e. its complex differentiability. To study the fact of the theory of analytic functions is the identity of the corresponding classes of functions in an arbitrary domain of the complex plane.

Keywords: Analytic Functions, Complex Analysis, Holomorphic and Homomorphic Function.

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INTRODUCTION:

Complex analysis, traditionally known as the theory of functions of a complex variable, is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatorics, applied mathematics; as well as in physics, including hydrodynamics and thermodynamics and also in engineering fields such as nuclear, aerospace, mechanical and electrical engineering.Complex analysis is particularly concerned with analytic functions of complex variables (or, more generally, meromorphic functions). Because the separate real and imaginary parts of any analytic function must satisfy Laplace's equation, complex analysis is widely applicable to two-dimensional problems in physics.

COMPLEX VARIABLES:

The complex number system is merely a logical extension of the real number system. The set of complex numbers includes the real numbers and still more. All complex numbers are of the form

$$Z= x + iy$$

where $i = \sqrt{-1}$. In other words $i^2 = -1$. If $y = 0$, then the complex number $x + iy$ becomes the real number x. This is why we say that the complex numbers are still "more" than the reals. The real numbers form a proper subset of the reals. We do not mean that the complex numbers are more numerous. We simply mean that they subsume the reals. Because there are two real numbers (x and y) associated with each complex number, we are able to depict complex numbers using a plane, as opposed to the reals which are depicted on a line. Unlike the real number system, complex numbers are not ordered. This means that it is not meaningful to say $z_1 < z_2$ in the complex number system, even though such a thing is posssible in the reals.

It is possible to define addition and multiplication of complex numbers in the following intuitive ways:

Addition: $(x_1+iy_1) + (x_2 + iy_2) = (x_1+x_2) + i(y_1+y_2)$

Multiplication: $(x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2+x_2y_1)$

The complex number 0 + i0 is the complex counterpart of zero in the reals. It is the complex additive identity. We will at times simply denote it as 0. The multiplicative identity is equal to 1 + i0, which we will at times denote as 1. A complex number can be written as z, so long as we understand that z = x + iy. It is possible to discuss subtracting and dividing complex numbers. For example,

$$z_1 - z_2 = (x_1+iy_1) + (-x_2 + i(-y_2)) = (x_1 - x_2) + i(y_1 - y_2)$$

$$\frac{z_1}{z_2} = (x_1 + iy_1)\left(\frac{x_1}{x_1^2 + y_1^2} - i\frac{y_1}{x_1^2 + y_1^2}\right) = 1$$

In addition to the basic operations of addition, subtraction, multiplication, and division, we can also perform more complicated operations – such as taking the square root.

$\sqrt{z_1} = a + ib$ where $(a+ib)(a+ib) = z_1 = x_1 + iy_1$.

THE CAUCHY-RIEMANN EQUATIONS AND COMPLEX DIFFERENTIATION

Suppose that we consider $f(z) = z^2$ and substitute into this z = x+iy. We can therefore write this function again in the following way:

$$f(z) = z^2 = F(x,y) = (x+iy)^2 = (x^2 - y^2) + i2xy = u(x,y) + iv(x,y)$$

where $u(x,y) = (x^2 - y^2)$ and where $v(x,y) = 2xy$. Now since z = x+iy, we know that $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ from which it follows that $\frac{\partial x}{\partial z} = \frac{1}{2}$ and $\frac{\partial y}{\partial z} = \frac{1}{2i}$. Now consider the complex derivative f '(z).

$$f'(z) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} = (2x - 2yi)(\frac{1}{2}) + (-2y + 2xi)(\frac{1}{2i})$$

This of course reduces to $f'(z) = 2z$ and the result agrees with the derivative computed in the previous section using limits. Now suppose that F(x,y) = u(x,y) + iv(x,y) is *any*

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differentiable complex function. What must be true about the functions u and v ? This is the subject of the Cauchy - Riemann equations. First, suppose that z changes by x changing alone. Then, assume that z changes by y changing alone. This would give us two expressions for the derivative of

f(z) = F(x,y).

The first (*holding y constant*) can be written as

$$f'(z)|_{y\ constant} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + i\frac{\partial v}{\partial x}\frac{\partial x}{\partial z} = \frac{1}{2}\{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\}$$

while the second (*holding x constant*) can be written as

$$f'(z)|_{x\ constant} = \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} + i\frac{\partial v}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\{-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\}.$$

Now, the derivative of f(z) cannot depend on which way that z is changing (either by x changing alone or alternatively by y changing alone) and so the two expressions for $f'(z)$ must be equal if the derivative exists. This implies that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and that} \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

These two equalities are known as the Cauchy-Riemann Equations.

ANALYTIC FUNCTIONS

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is analytic at a point z_0 if it has a derivative at each point in some neighborhood of z_0. It follows that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0. A function f is analytic in an open set if it has a derivative everywhere in that set. If we should speak of a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S. Note that the function $f(z) = \frac{1}{z}$ is analytic at each nonzero point in the finite plane.

But the function $f(z) = |z|^2$ is not analytic at any point since its derivativeexists only at $z = 0$ and not throughout any neighborhood. An entire function is a function that is analytic at each point in the entire finite plane. Since the derivative of a polynomial exists everywhere, it follows that every polynomial is an entire function. If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, thenz_0 is called a singular point, or singularity, of f. The point $z = 0$ is evidently a singular point of the function $f(z) = \frac{1}{z}$. The function $f(z) = |z|^2$, on the other hand, has no singular points since it is nowhereanalytic.

Series (Taylor's Series)

We turn now to Taylor's theorem, which is one of the most important results of this section.

Taylor Series Generated by *f* at *x* = *a*

Let f be a function with derivatives o all orders throughout some open interval containing a. Then the **Taylor series generated by *f* at *x* = *a*** is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum \frac{f^{(k)}(a)}{k!}(x-a)^k. \text{ The}$$

partial sum

$$P_n(x) = \sum_{k=o}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Is the **Taylor polynomial of order *n* for *f* at *x* = *a***.

Taylor's Theorem with Remainder

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{n+1}c}{(n+1)!}(x-a)^{n+1}$$

for some *c* between *a* and *x*.

Remainder Estimation Theorem

If there are positive constants *M* and *r* such that $\left| f^{n+1}t \right| \leq Mr^{n+1}$ for all *t* between *a* and *x*, then the remainder $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$\left| R_n(x) \right| \leq M \frac{r^{n+1}\left| x-a \right|^{n+1}}{(n+1)!}.$$

If these conditions hold for every *n* and all other conditions of aylor's Theorem are satisfied by *f*, then the series converges to *f*(*x*).

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdot \ (|z - z_0) < R_0). \ (1.2.7.3)$$

Any function which is analytic at a point z_0 must have a Taylor series about z_0. For, if f is analytic at z_0, it is analytic throughout some neighborhood $|z - z_0| < \varepsilon$ of that point ; and ε may serve as the value of R_0in the statement of Taylor's theorem. Also, if$f$ is entire,R_0 can be chosen arbitrarily large; and the condition of validity becomes $|z - z_0| < \infty$. The series then converges to $f(z)$ at each point$z$in the finite plane. When it is known that f is analytic everywhere inside a circle centered atz_0, convergence of its Taylor series about z_0 to $f(z)$ for each point z within that circle is ensured; no test for the convergence of the series is even required. In fact, according to Taylor's theorem, the series converges to $f(z)$ within the circle about z_0 whose radius is the distance from z_0 to the nearest point $z_1$at which f fails to be analytic. we shall find that this is actually the largest circle centered at z_0 such that the series converges to $f(z)$ for allz interior to it. In the following section, we shall first prove Taylor's theorem when $z_0 = 0$, in which case f is assumed to be analytic throughout a disk $|z| < R_0$ and series (1.7.2.1) becomes a Maclaurin series:

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$$f(z) = \sum_{n=0}^{\infty} (\frac{f^{(n)}(0)}{n!})\, z_n \; (|z| \; < R_0). (1.7.2.4)$$

HOLOMORPHIC FUNCTIONS:

Holomorphic functions is a topic that involves differential equations and the complex plane. Its applications are diverse and the properties of these functions are interesting. There are three interesting cases for singular points arising from these functions. The complex plane **C**, is the basis in which holomorphic functions are defined. One axis in the complex plane is the real axis on which all real numbers lie. The other axis is the imaginary axis on which the imaginary numbers lie. Imaginary numbers are characterized by a number multiplied by the square root of negative one, *i*.

A holomorphic function is defined to be a differentiable complex function with a continuous derivative. Some books define a holomorphic function without the continuous derivative condition.. Holomorphic functions can be represented as power series inside the circle of convergence. Kodaira 84 contains many prrofs to show that basic properties such as the chain rule for integration, addition, and so forth for holomorphic functions are valid on **C**. Cauchy's Integral Formula yields: since the above exists, and the following results after some manipulation:

which is the Mean Value Theorem for these functions where *i* = sqrt(-1), z is in **C**, w = c where c is a point, and The nth derivatives of a holomorphic function are holomorphic. If $f_n(z)$ is a sequence comprised of holomorphic functions in the region D and they converge on the compact subsets of D, then the limit is holomorphic on D. It is also useful to realize that if the continuous partial derivatives $f_x(z)$ and $f_y(z)$ exist in D and if they also satisfy the Cauchy-Riemann equations in D, then f(z) is holomorphic on D.

Singularities often arise when f(z) is represented as a power series as follows:

This power series converges absolutely on an interval U. The above is known as a Laurent series expansion of *f*(z) about the point c. The portion of the Laurent expansion from n = -infinity to n

= -1 is known as the principal part of the expansion. Three possibilities exist with regard to singularities in holomorphic functions.

If there is no principal part, then we set *f*(c) = a_0 in order to remove the singularity. So, c is a removable singularity of *f*(z). If the principal part contains a number of finite terms, then c is a pole of *f*(z) of order m depending on the quantity $(z - c)^m$.. In this situation poles and zeros exist, which can be dealt with using standard methods. The third possibility is that the principal part is an infinite series. This is known as an essential singularity. Weierstrass' Theorem applies: If c is a point at an essential singularity and w is in **C** for f(z), then one can find a sequence of z_n points that converge to c so that the limit:

The analysis connection to differential equations shows up everywhere. Harmonic motion in the real world is the most classic example which can be represented in **C**. The wave equation, the spherical Bessel differential equation, the Helmholtz differential equation, and many other highly-useful named equations operate regularly in the realm of holomorphic functions. The field of electrostatics is another interesting application of these functions.. The three possibilities above and their associated theorems simply provide a way to understand and better approximate singularities in holomorphic functions. For an engineer, these functions show up regularly when studying frequency response where the neper frequency forms the real axis and the angular frequency forms the imaginary axis. Understanding these functions is thus of extreme importance to engineers who don't like things to either blow up or not work at all.

DERIVATION:

A function *f*: D → C defined on an open set D in the complex plane is said to be holomorphic on D if the limit

$$f^{'}(z) = \lim_{h \to o} \frac{f(z\ +\ h)\ - f(z)}{h}$$

is defined for all z 2 D. The value of this limit is denoted by *f'*(z), or by df(z)/dz, and is referred to as the derivative of the function f at z.

Note that if *f*: D → C is a holomorphic function defined on an open set C in the complex plane then f is continuous on D. For let z ϵ D. Then

$$\lim_{h\to 0} f'(z + h) = \lim_{h\to 0}(f(z) + hx\,(\frac{f(z + h) - f(z)}{h}))$$

$$= f(z) + (\lim_{h\to 0} h)(\lim_{h\to 0} \frac{(f(z + h) - f(z))'}{h}$$

$$= f(z)$$

and thus the function f is continuous at z, as required.

Lemma 3.1 A function *f*: D → C, defined on an open set D in the complex plane, is holomorphic on D if and only if, given any complex number w belonging to D, and given any positive real number ϵ, there exists some real positive number δ such that $\left|f(z) - f(w) - (z - w)f'(w)| \le \epsilon|z - w|\right|$.

Proof: The function f has a well-defined derivative f0(w) at a point w of D if and only if

$$f'(w) = \lim_{h\to 0} \frac{f(w + h) - f(w)}{h} = \lim_{z\to w} \frac{f(z) - f(w)}{z - w}$$

This limit exists if and only if, given any positive real number ", there exists some positive real number δ such that

$$\left|\frac{f\left(z - f(w) - f(z - w)f'(w)\right)}{z - w}\right| \le \varepsilon$$

Whenever 0 < |z−w| < δ. The required result follows directly on rearranging the above inequality. Proposition 3.1: Let f:D ! C and g:D ! C be holomorphic functions defined over an open set D in the complex plane. Then the sum f + g, difference f − g and product f · g of the functions f and g are holomorphic, where (f + g)(z) = f(z) + g(z), (f − g)(z) = f(z) − g(z) and (f · g)(z) = f(z)g(z) for all z ϵ D. Moreover (f + g)(z) = f'(z) + g(z), (f − g)(z) =f(z) – g(z) and (f · g)(z) = f(z)g(z) + f(z)g(z) for all z ϵ D.

Proof: The results for f + g and f − g follow easily from the fact that the limit of a sum or difference of two complex-valued functions is the sum or difference of the limits of those functions. The limit of a product of complex-valued functions is the product of the limits of those functions, and therefore

$$\begin{aligned}(f\cdot g)'(z) &= \lim_{h\to 0}\frac{f(z+h)g(z+h)-f(z)g(z)}{h}\\ &= \lim_{h\to 0}\left(\frac{(f(z+h)-f(z)}{h}g(z+h)\right)\\ &\quad+\lim_{h\to 0}\left(f(z)\frac{g(z+h)-g(z)}{h}\right)\\ &= f'(z)g(z)+f(z)g'(z),\end{aligned}$$

Lemma 3.2: Let $f:U\to\mathbb{C}$ be continuous and $f=u+iv$. Then $f\in\mathcal{H}(U)$ if and only if u_x, u_y, v_x, v_y all exist and satisfy the **Cauchy-Riemann equations**:

$$u_x = v_y \text{ and } u_y = -v_x.$$

Proof: First assume f is holomorphic. Then certainly all of the partials exist. We can see, for instance, that $u_x = \operatorname{Re}(f')$. Let $z_0 = x_0 + iy_0 \in U$. If we take the limit as $z \to z_0$ by taking $z = z_0 + t$ for $t \in \mathbb{R}$ going to zero, then

But we could also take take z to z_0 by approaching from another direction. I could approach along the imaginary axis, taking $z = z_0 + it$, and letting $t \in \mathbb{R}$ go to zero. From this direction, we get that

These give two different expressions for f', and so we can equate them. Taking the real parts of each tells us that $u_x = v_y$. Taking the imaginary parts of each tells us that $v_x = -u_y$.

In the other direction, let $w = a + bi$. From the existence of the partials, we know

$$\frac{u(z_0+w)-u(z_0)-u_x(z_0)a-u_y(z_0)b}{w}\to 0$$

as $w \to 0$. Similarly, for v, we have that

$$\frac{v(z_0 + w) - v(z_0) - v_x(z_0)a - v_y(z_0)b}{w} \to 0$$

as $w \to 0$. Now what we want is to find some L for which $\lim_{w \to 0} \frac{f(z_0 + w) - f(z_0)}{w} = L$. Let's try $L = u_x(z_0) + iv_x(z_0)$, because we used it in the first half of the proof. If anything should work, that should. We wish to show that

$$\frac{f(z_0 + w) - f(z_0) - L \cdot w}{w} \to 0$$

as $w \to 0$. So consider just the real part of the numerator. From the Cauchy-Riemann equations, $v_x(z_0) = -u_y(z_0)$. Then, from the existence of the partials, we have

$$\begin{aligned}\frac{\mathrm{Re}(f(z_0 + w) - f(z) - w \cdot L)}{w} &= \frac{u(z_0 + w) - u(z_0) - u_x(z_0)a + v_x(z_0)b}{w} \\ &= \frac{u(z_0 + w) - u(z_0) - u_x(z_0)a - u_y(z_0)b}{w} \to 0.\end{aligned}$$

As one might expect, something nearly identical happens when we take the just the imaginary part of the numerator.

$$\begin{aligned}\frac{\mathrm{Im}(f(z_0 + w) - f(z) - w \cdot L)}{w} &= \frac{v(z_0 + w) - v(z_0) - u_y(z_0)a - v_y(z_0)b}{w} \\ &= \frac{v(z_0 + w) - v(z_0) - v_x(z_0)a - v_y(z_0)b}{w} \to 0.\end{aligned}$$

Putting these together gives the desired result.

HARMONIC FUNCTION:

The study a certain functions defined on subsets of the complex plane which are real valued. The main motivation for studying them is that the partial differential equation they satisfy is very common in the physical sciences.

Definition: Let G ⊆ C be a region. A function u : G ! R is harmonic in G if it has continuous second partials in G and satisfies the Laplace1 equation in G.

$U_{xx} + U_{yy} = 0$

There are (at least) two reasons why harmonic functions are part of the study of complex analysis, and they can be found in the next two theorems.

Proposition 3.1: Suppose f = u+iv is holomorphic in the region G. Then u and v are harmonic in G.

Proof. First, by Corollary 5.2, f is infinitely differentiable, and hence so are u and v. In particular, u and v have continuous second partials. By Theorem 2.13, u and v satisfy the Cauchy Riemann equations

$u_x = v_y$ and $u_y = -v_x$

in G. Hence

$u_{xx} + u_{yy} = (u_x)x + (u_y)y = (v_y)x + (-v_x)y = v_{yx} - v_{xy} = 0$

in G. Note that in the last step we used the fact that v has continuous second partials. The proof that v satisfies the Laplace equation is completely analogous.

Theorem 3.2: Suppose u is harmonic on the simply connected region G. Then there exists a harmonic function v such that f = u + iv is holomorphic in G. Remark. The function v is called a harmonic conjugate of u.

Proof. We will explicitly construct the holomorphic function f (and thus v = Im f). First, let

$g = u_x - iu_y$:

The plan is to prove that g is holomorphic, and then to construct an antiderivative of g, which will be almost the function f that we're after. To prove that g is holomorphic, we use Theorem 3.2: first because u is harmonic, Re g = u_x and Im g = $-u_y$ have continuous partials. Moreover, again because u is harmonic, they satisfy the Cauchy{Riemann equations:

$$(\operatorname{Re} g)_x = u_{xx} = -u_{yy} = (\operatorname{Im} g)_y$$

And

$$(\operatorname{Re} g)_y = u_{xy} = u_{yx} = -(\operatorname{Im} g)_x$$

Now that we know that g is holomorphic in G, we can use Theorem 5.14 to obtain a primitive h of g on G. (Note that for the application of this theorem we need the fact that G is simply connected.) Suppose we decompose h into its real and imaginary parts as

h = a + ib. Then, again using Theorem 3.2,

$$g = h' = a_x + ib_x = a_x - ia_y$$

(The second equation follows with the Cauchy{Riemann equations.) But the real part of g is u_x, so that we obtain $u_x = a_x$ or u(x; y) = a(x; y) + c(y) for some function c which only depends on y. On the other hand, comparing the imaginary parts of g and h_0 yields

$- u_y = - a_y$ or u(x; y) = a(x; y) + c(x),

and c depends only on x. Hence c has to be constant, and u = a + c. But then

f = h + c

is a function holomorphic in G whose real part is u, as promised.

Remark. In hindsight, it should not be surprising that the function g which we first constructed is the derivative of the sought-after function f. Namely, by Theorem 3.2 such a function f = u+iv must satisfy

$f' = u_x + iv_x = u_x - iu_y$:

(The second equation follows with the Cauchy{Riemann equations.) It is also worth mentioning that the proof shows that if u is harmonic in G then ux is the real part of a function holomorphic in G regardless whether G is simply connected or not.

As one might imagine, the two theorems we've just proved allow for a powerful interplay between harmonic and holomorphic functions. In that spirit, the following theorem might appear not too surprising. It is, however, a very strong result, which one might appreciate better when looking back at the simple definition of harmonic functions.

A harmonic function is infinitely differentiable.

Proof. Suppose u is harmonic in G. Fix z_0 2 G and r > 0 such that the disk

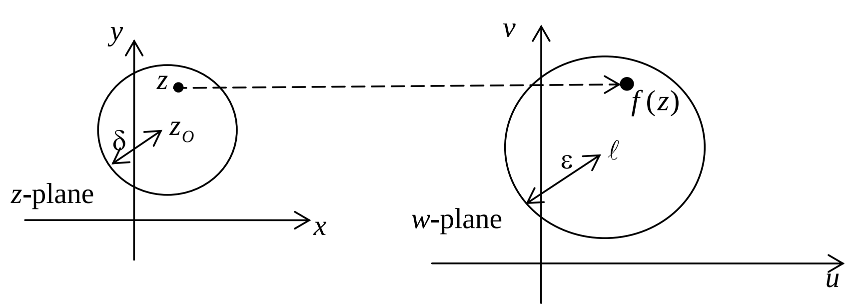
$$D = \{z \in \mathbb{C} : |z - z_0| < r\}$$

is contained in G. D is simply connected, so by the last theorem, there exists a function *f* holomorphic in D such that u = Re f on D. *f* is infinitely differentiable on D, and hence so is its real part u. Because z_0 2 D, we showed that u is infinitely differentiable at z_0, and because z_0 was chosen arbitrarily, we proved the statement.

Remark. This is the first in a series of proofs which uses the fact that the property of being harmonic is a local property| it is a property at each point of a certain region. Note that we did not construct a function f which is holomorphic in G but we only constructed such a function on the disk D.

APPLICATIONS

We need to define the concepts of limit, continuity and differentiability for functions of a complex variable. A function *f*(*z*) has the limit ℓ as $z \to z_0$, if for any real $\varepsilon > 0$, $\exists$ real $\delta > 0$ such that for all $z \neq z_0$ such that $|z - z_0| < \delta$, $|f(z) - \ell| < \varepsilon$, we write $\lim_{z \to z_0} f(z) = \ell$ or $f(z) \to \ell$ as $z \to z_0$. Geometrically, this means that *f*(*z*) must lie within an open disc with centre ℓ and arbitrary small radius ε whenever *z* lies within an open disc of radius δ and with centre z_0



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Note that z may approach z_0 from any direction and is hence more restrictive than the equivalent definition for a function $f(x)$ of a real variable. Indeed it is closely related to the definition for a function of two real variables.

CAUCHY'S INTEGRAL THEOREM

Cauchy's integral theorem must also be slightly modified in the spatial case, and is then known as the Cauchy–Poincaré theorem: Let a function f be holomorphic in a domain D; then, for any compact n+1-dimensional (real!) submanifold G in D with piecewise-smooth boundary ∂G,

$$\int_{\partial G} f(z)dz = 0.$$

Here the integral is the integral of the differential form $\omega=f(z)\, dz_1\wedge \ldots \wedge dz_n$, which in the real variables $x_1,y_1,\ldots,x_n,y_n$ can be simply written as $\omega_r+i\omega_i$, where ω_r and ω_i are *real* differential forms.

As in the planar case, this integral can be defined by a parametric representation of the given set: if $z:\mathrm{R}^n\supset\Omega\to\partial G$ is a parametrization of a portion A of ∂G, then

$$\int_{\partial G} f(z)dz = 0\,.$$

where

$\partial(z_1,\ldots,z_n)\partial(t_1,\ldots,t_n)$

denotes the determinant of the $n\times n$ complex matrix

$\partial z_i \partial t_j$.

The integral can then be defined using charts.

Observe that, in contrast with the case n=1, when n>1 the dimension of the surface G (which is n+1) is strictly smaller than the (topological) dimension of the ambient domain D (which is 2n).

Observe also that can be concluded from Stokes theorem: the holomorphy of *f* implies in fact that $d(fdz_1,\dots dz_n)=0$.

CAUCHY'S THEOREM:

Cauchy's theorem, also called the Cauchy-Goursat theorem. If f(z) is analytic (that is, point-wise differentiable) on a simply connected open region D⊂C and if C is any rectifiable closed contouror cycle in D , then

$$\int_C f(z)\,dz = 0.$$

Our proof of Cauchy's theorem will proceed in several steps.

We begin by establishing Cauchy's theorem for triangles. Another possible starting point is rectangles with sides parallel to the axes. There are in fact several quite distinct ways to arrive at the generality in the statement of Cauchy's theorem just given, and of the extensions to follow. Some of these alternative approaches will be examined briefly.Recall that, at this stage, analytic just means point wise differentiable on a region. The additional smoothness that complex analytic functions necessarily possess is the subject of theorems that follow on after Cauchy's theorem.The curve C is not necessarily simple. It may self-intersect in any manner, provided it is closed and rectifiable. Rectifiable includes piecewise smooth, which is the class of contours of greatest interest in complex analysis. Rectifiable paths are more natural for real-variable line integrals, which makes them more natural to work with in some of the proofs in the complex domain. Nevertheless, most textbooks restrict attention to piecewise smooth paths, but their proofs are usually easy to adapt to the rectifiable case. Some also restrict attention to simple closed curves, which is a more significant loss of generality. Generalisations to domains that are not simply connected and to functions that may not be analytic will be considered after the proof of Theorem.

Cauchy's original proof employed Green's theorem in the plane. He needed to impose stricter hypotheses on f(z) and C than are needed in Goursat's later proof. Suppose, for the purposes of this paragraph only, that f′(z) is continuous in D and that C is piecewise smooth and simple. Let E denote the inside of C plus C itself. Use the same symbols C,

D and E for the corresponding sets in the real xy-plane. Put

z=x+iy and f(z) =u(x,y) +iv(x,y). Under these hypotheses,u and v satisfy the Cauchy-Riemann equations, ux=vyand uy=−vx, the partial derivatives being continuous in D

in the present context. Then, two applications of Green's theorem give

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\\ &= -\iint_E (v_x + u_y)\,dx\,dy + i\iint_E (u_x - v_y)\,dx\,dy\\ &= 0.\end{aligned}$$

This quick proof appears in many textbooks, and is frequently done in lectures when the purpose is to get through Cauchy's theorem quickly on the way to applications.The quick proof is unsatisfactory for a couple of reasons. The obvious reason is that we have imposed unnecessary restrictions on C and f(z). A more potent objection is that Green's theorem is at least at the same mathematical depth as Cauchy's theorem, and so one should not claim a proof of one on the basis of the other unless the other has been proved rigorously under decent hypotheses. The proof of Green's theorem for rectifiable Jordan curves, for example, is not easy and was first done by D. H. Potts (1951). The latter proof is included in the first edition of TomM. Apostol:Mathematical Analysis. Also Green's theorem is poorly suitedto self-intersecting contours.

Edouard Goursat (1900) proved Cauchy's theorem for piecewise smooth Jordan curves by partitioning the interior into small squares, with residual fragments next to the boundary. He did not require f′(z) to be continuous. Ahlfors' textbook Complex Analysis

contains a proof of a general form of Cauchy's theorem (Theorem 3.18 below) that includes a stage where are region is partitioned in this fashion. Alfred Pringsheim (1901) simplified and strengthened Goursat's proof by starting with the special case where Cis a triangle. This approach allows for upgrades in stages to self-intersecting rectifiable contours.

Some textbooks begin by proving Cauchy's theorem for a circular or rectangular contour. From there, they deduce Cauchy's integral formula for circular or rectangular contours, which then implies the differentiability of f (z) to all orders, the Cauchy-Taylor theorem (circular contours preferred), and many of the incidental theorems and corollaries that we will meet in the coming pages, but with the contours restricted. Having obtained all this extra information about analytic functions in simple domains, they can then turn their attention to more general contours and more complicated domains of analyticity.Another approach is to prove that an analytic function in an open disc has a primitive function. This can be deduced from a proof of Cauchy's theorem for rectangles with sides parallel to the axes. A primitive function implies Cauchy's theorem for arbitrary rectifiable closed contours according to Theorem 2.12, but the domain D is restricted to a disc. The next step is to prove Cauchy's integral formula for Jordan curves in a disc and use it to step up to more general formulations of Cauchy's theorem.

Pringsheim's lemma (Cauchy's theorem for triangles). Suppose that f(z) is analytic in a region D. Let Δ be any triangle in D whose interior is also contained in D. Then

$$\int_{\Delta} f(z)\, dz = 0.$$

Proof

Suppose, on the contrary, that

$$I_0 := \int_{\Delta} f(z)\, dz \neq 0.$$

Let A,B and C denote the vertices of Δ in the order following its orientation and let L,M and N denote the midpoints of the sides AB, BC and CA, respectively. Form the four smaller oriented congruent triangles, δ_{01}=ALN,δ_{02}=LBM,δ_{03}=NMC and δ_{04}=MNL. Then the internal edges are followed in both directions and so

$$\Delta = ABC = \sum_{k=1}^{4} \delta_{0k} ,$$

$$I_0 = \sum_{k=1}^{4} \int_{\delta_{0k}} f(z)\, dz.$$

Let *I1* denote the integral on the right-hand side with largest modulus and let Δ1denote the corresponding triangle (one of theδ_{0k}). If two or more integrals share the same largest modulus, choose the one with lowest k. Then, according to the triangle inequality for complex numbers,

$$|I_0| \le \sum_{k=1}^{4} \left| \int_{\delta_{0k}} f(z)\, dz \right| \le 4|I_1|.$$

Next, the triangle Δ_1 can be subdivided in the same manner into four congruent triangles

δ_{1k},k= 1,2,3,4, and one of those triangles Δ_2, around which the integral of f(z) is I2, can be selected so that

$$|I_1| \le 4|I_2|.$$

This process can be iterated indefinitely. The sequence of integrals,I_0,I_1,I_2,..., around the respective triangles, $\Delta_0 := \Delta$, Δ_1, Δ_2,..., satisfies

$$|I_{n-1}| \le 4|I_n|, \qquad |I_0| \le 2^{2n}|I_n|.$$

f the perimeter of Δ is L_0, the perimeter of Δ_n is Ln= $2^{-n}L_0$. Let En denote the closed set consisting of the triangle Δn and its interior. The sequence of closed sets,E_0,E_1,E_2,..., satisfies

E_0, E_1, E_2, …………….E_n... .

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There is exactly one limit point z_0 common to all of the E_n. Because $z_0 \in \Delta$, it is also contained in D.We now make use of the differentiability of f(z) at z_0 . Given ϙ > 0, there exists δ >0 such that, for all z in the open disc $|z-z_0|< \delta$

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0)$$

With $|\eta(z)|<$ϙ. For sufficiently largen, the triangle Δ n and all later triangles in the equence are contained within the open disc $|z-z_0|< \delta$. Since z0is inside or on Δn, this triangle will always be inside the disc if $L_n<2\delta$. Now

$$\begin{aligned} I_n &= \int_{\Delta_n} f(z)\,dz \\ &= (f(z_0) - z_0 f'(z_0)) \int_{\Delta_n} 1\,dz + f'(z_0)\int_{\Delta_n} z\,dz + \int_{\Delta_n} \eta(z)(z - z_0)\,dz \\ &= \int_{\Delta_n} \eta(z)(z - z_0)\,dz, \end{aligned}$$

Where we used Lemma 2.11 to set the integrals of 1 and z to zero. In the last integrand, we have the bounds, $|\eta(z)| < \epsilon$ and $|z - z_0| < \frac{1}{2}L_n$, for $z \in \Delta_n$. Hence, according to the ML formula

$$|I_n| = \left|\int_{\Delta_n} \eta(z)(z - z_0)\,dz\right| \le \tfrac{1}{2}\epsilon(L_n)^2 = 2^{-2n-1}\epsilon(L_0)^2.$$

$$|I_0| \le 2^{2n}|I_n| \le \tfrac{1}{2}\epsilon(L_0)^2.$$

Because ϙis arbitrary, the right-hand side is arbitrarily small. This contradicts our initial assumption that $I_{06} = 0$. HenceI_0= 0. In other words,

$$\int_{\Delta} f(z)\,dz = 0$$

under the stated hypotheses.

CONCLUSION

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The study concludes the complex function is said to be analytic on a region R if it is complex differentiable at every point in R. If a complex function is analytic on a region R, it is infinitely differentiable in R. A complex function may fail to be analytic at one or more points through the presence of singularities, or along lines or line segments through the presence of branch cuts. A complex function that is analytic at all finite points of the complex plane is said to be entire. The study of harmonic functions originally arose from physics but our interest in them stems from the fact that the real and imaginary parts of holomorphic functions are harmonic. The study reviewsthe basic properties of harmonic functions in this chapter. For a more extensive treatment, check any good textbook on potential theory.The requirement that f be holomorphic on $U_0 = U \setminus \{a_k\}$ is equivalent to the statement that the exterior derivative $d(fdz) = 0$ on U_0. Thus if two planar regions V and W of U enclose the same subset $\{a_j\}$ of $\{a_k\}$, the regions $V \setminus W$ and $W \setminus V$ lie entirely in U_0.

The study of Analytic funciton can be used in widely applied like heat conduction, fluid flows. Electro statics is real-valued integrals are sometimes easily, solvable by complexification. The idea of an analytic function includes the whole wealth of functions, most important to science, whether they have their origin in number theory, in the theory of differential equations or of Algebric Functional Equatioins and arise in mathematical physics.Find also the study of harmonic functions originally arose from physics but our interest in them stein from the fact that the real imagery parts of harmorphicfunciton are harmonic.Analysis of Analytic functions and complex analytic funciton categories that are similar are some ways, but different in others. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not hold generally for real analytic functions and we see the relationship of the residue theorems to stokes theorem is given by the Jordan Curve Theorem.

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