Solving Integral Problems Using Two Theorems of Cauchy

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Abstract:

In this paper, we study two types of definite integrals. The closed forms of the two types of definite integrals can be determined by using two theorems of Cauchy. Moreover, some examples are provided to do calculation practically. The research method adopted in this study is to find solutions through manual calculations and verify our answers using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Keywords

Definite Integrals, Closed Forms, Two Theorems of Cauchy, Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights.

In calculus and engineering mathematics, there are many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. This article considers the following two types of definite integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^{2\pi} \frac{\sum_{n=0}^{m} \sum_{p=0}^{k+1} \frac{(k+1)_P}{p!} a_n r^{n+k+1-P} (-s)^P \cos[(n-k+p)\theta - p\varphi]}{[r^2 - 2rs\cos(\theta-\varphi) + s^2]^{k+1}} d\theta,$$

(1)

$$\int_0^{2\pi} \frac{\sum_{n=0}^{m} \sum_{p=0}^{k+1} \frac{(k+1)_P}{p!} a_n r^{n+k+1-P} (-s)^P \sin[(n-k+p)\theta - p\varphi]}{[r^2 - 2rs\cos(\theta-\varphi) + s^2]^{k+1}} d\theta,$$

(2)

where r, s, θ, φ are real numbers, m, k are non-negative integers, $|s| \neq |r|$, and a_n are real numbers for all non-negative integers n ($n \leq m$). The closed forms of the two types of definite integrals can be obtained using Cauchy theorem and Cauchy integral formula for derivatives; these are the major results of this paper (i.e., Theorems 1 and 2). Yu [1-26], Yu and B. -H. Chen [27], and Yu and Sheu [28-30] used the following techniques: complex power series method, integration term by term theorem, differentiation with respect to a parameter, Parseval's theorem, and area mean value theorem, to evaluate some types of integral problems. In this paper, two examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

2. Methods and Results

Firstly, we introduce some notations, formulas and theorems used in this paper.

2.1. Notations:
Define $(r)_q = r(r-1)\cdots(r-q+1)$, and $(r)_0 = 1$, where r is a real number, and q is a positive integer.

2.2. Formulas and theorems:

2.2.1. Euler's formula:

$e^{ix} = \cos x + i \sin x$, where $i = \sqrt{-1}$, and x is any real number.

2.2.2. DeMoivre's formula:

$(\cos x + i \sin x)^m = \cos mx + i \sin mx$, where m is any integer, and x is any real number.

2.2.3. Binomial theorem:
Suppose that u, v are complex numbers, and n is a positive integer, then $(u+v)^n = \sum_{m=0}^{n} \frac{(n)_m}{m!} u^{n-m} v^m$.

Next, we introduce two theorems of Cauchy in complex analysis used in this study, which can be found in [31, p109] and [32, p115] respectively.

2.2.4. Cauchy theorem:
Let f be analytic on a simply connected region D and C be a simple closed curve in D, then $\int_C f(z)dz = 0$.

2.2.5. Cauchy integral formula for derivatives:
Suppose that f is an analytic function in a simply connected domain D and that C is a regular closed curve lying in D. Then for each z inside C and $k = 0,1,2,\cdots$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{k+1}} dw.$$

In the following, we determine the closed form of the definite integral (1).

Theorem 1 *Suppose that r, s, θ, φ are real numbers, m, k are non-negative integers, $|s| \neq |r|$, and a_n are real numbers for all non-negative integers $n \leq m$, then the definite integral*

$$\int_0^{2\pi} \frac{\sum_{n=0}^{m}\sum_{p=0}^{k+1} \frac{(k+1)_p}{p!} a_n r^{n+k+1-p}(-s)^p \cos[(n-k+p)\theta - p\varphi]}{[r^2 - 2rs\cos(\theta - \varphi) + s^2]^{k+1}} d\theta$$

$$= \begin{cases} 0 & if\,|s| > |r| \quad (3) \\ \frac{2\pi}{k!\,r}\sum_{n=0}^{m}(n)_k a_n s^{n-k}\cos(n-k)\varphi & if\,|s| < |r| \quad (4) \end{cases}$$

Proof Let $w = re^{i\theta}$, $z = se^{i\varphi}$, $f(z) = \sum_{n=0}^{m} a_n z^n$, and C be a positively oriented circle with center at 0 with radius $|r|$, then

$$\frac{k!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{k+1}} dw$$

$$= \frac{k!}{2\pi i} \int_C \frac{\sum_{n=0}^{m} a_n w^n}{(w-z)^{k+1}} dw$$

$$= \frac{k!\,r}{2\pi} \int_0^{2\pi} \frac{\sum_{n=0}^{m} a_n r^n e^{i(n+1)\theta}}{(re^{i\theta} - se^{i\varphi})^{k+1}} d\theta$$

(by Euler's formula and DeMoivre's formula)

$$= \frac{k!\,r}{2\pi} \int_0^{2\pi} \frac{\sum_{n=0}^{m} a_n r^n e^{i(n+1)\theta}(re^{-i\theta} - se^{-i\varphi})^{k+1}}{[r^2 - 2rs\cos(\theta - \varphi) + s^2]^{k+1}} d\theta$$

$$= \frac{k!\,r}{2\pi} \int_0^{2\pi} \frac{\sum_{n=0}^{m} a_n r^n e^{i(n+1)\theta} \cdot \sum_{p=0}^{k+1} \frac{(k+1)_p}{p!}(re^{-i\theta})^{k+1-p}(-se^{-i\varphi})^p}{[r^2 - 2rs\cos(\theta - \varphi) + s^2]^{k+1}} d\theta$$

(by binomial theorem)

$$= \frac{k!\,r}{2\pi} \int_0^{2\pi} \frac{\sum_{n=0}^{m}\sum_{p=0}^{k+1} \frac{(k+1)_p}{p!} a_n r^{n+k+1-p}(-s)^p e^{i[(n-k+p)\theta - p\varphi]}}{[r^2 - 2rs\cos(\theta - \varphi) + s^2]^{k+1}} d\theta. \quad (5)$$

Case 1. If $|s| > |r|$. Since $r = 0$ is trivial, we may assume that $r \neq 0$. By Cauchy theorem, we have

$$\int_C \frac{f(w)}{(w-z)^{k+1}} dw = 0. \quad (6)$$

Using Eq. (5) yields

$$\int_0^{2\pi} \frac{\sum_{n=0}^{m}\sum_{p=0}^{k+1} \frac{(k+1)_p}{p!} a_n r^{n+k+1-p}(-s)^p e^{i[(n-k+p)\theta - p\varphi]}}{[r^2 - 2rs\cos(\theta - \varphi) + s^2]^{k+1}} d\theta = 0. \quad (7)$$

Thus, by the equality of real parts of both sides of Eq. (7), we obtain

$$\int_0^{2\pi} \frac{\sum_{n=0}^{m}\sum_{p=0}^{k+1} \frac{(k+1)_p}{p!} a_n r^{n+k+1-p}(-s)^p \cos[(n-k+p)\theta - p\varphi]}{[r^2 - 2rs\cos(\theta - \varphi) + s^2]^{k+1}} d\theta = 0.$$

Case 2. If $|s| < |r|$. Then using Cauchy integral formula for derivatives yields

$$\frac{d^k}{dz^k} \sum_{n=0}^{m} a_n z^n = \frac{k!}{2\pi i} \int_C \frac{\sum_{n=0}^{m} a_n w^n}{(w-z)^{k+1}} dw. \quad (8)$$

By Eq. (5), we have

$$\int_0^{2\pi} \frac{\sum_{n=0}^{m}\sum_{p=0}^{k+1} \frac{(k+1)_p}{p!} a_n r^{n+k+1-p}(-s)^p e^{i[(n-k+p)\theta - p\varphi]}}{[r^2 - 2rs\cos(\theta - \varphi) + s^2]^{k+1}} d\theta$$

$$= \frac{2\pi}{k!\,r} \sum_{n=0}^{m}(n)_k a_n s^{n-k} e^{i(n-k)\varphi}. \quad (9)$$

Therefore, using the equality of real parts of both sides of Eq. (9) yields

$$\int_0^{2\pi} \frac{\sum_{n=0}^{m}\sum_{p=0}^{k+1} \frac{(k+1)_p}{p!} a_n r^{n+k+1-p}(-s)^p \cos[(n-k+p)\theta - p\varphi]}{[r^2 - 2rs\cos(\theta - \varphi) + s^2]^{k+1}} d\theta$$

$$= \frac{2\pi}{k!\,r} \sum_{n=0}^{m}(n)_k a_n s^{n-k} \cos(n-k)\varphi.$$

q.e.d.

Next, we determine the closed form of the definite integral (2).

Theorem 2 *Assume that the assumptions are the same as Theorem 1, then*

$$\int_0^{2\pi} \frac{\sum_{n=0}^{m} \sum_{p=0}^{k+1} \frac{(k+1)_p}{p!} a_n r^{n+k+1-p}(-s)^p \sin[(n-k+p)\theta - p\varphi]}{[r^2 - 2rs\cos(\theta-\varphi) + s^2]^{k+1}} d\theta$$

$$= \begin{cases} 0 & if\,|s| > |r| \quad (10) \\ \frac{2\pi}{k!\,r} \sum_{n=0}^{m}(n)_k\, a_n s^{n-k} \sin(n-k)\varphi & if\,|s| < |r| \quad (11) \end{cases}$$

Proof If $|s| > |r|$, then by the equality of imaginary parts of both sides of Eq. (7), we obtain Eq. (10). If $|s| < |r|$, then using the equality of imaginary parts of both sides of Eq. (9) yields Eq. (11) holds.

q.e.d.

3. Examples

In the following, for the two types of definite integrals in this paper, we propose two examples and use Theorems 1 and 2 to obtain their closed forms. In addition, we use Maple to calculate the approximations of these definite integrals and their solutions for verifying our answers.

3.1. Example

In Theorem 1, let $r = -3, s = 4, \varphi = \pi/3, m = 1,$, $a_0 = 2,$ and $a_1 = -5$, then by Eq. (3) we obtain

$$\int_0^{2\pi} \frac{159 + 18\cos\theta + 32\cos\left(\theta - \frac{2\pi}{3}\right) + 360\cos\left(\theta - \frac{\pi}{3}\right) + 240\cos\left(2\theta - \frac{2\pi}{3}\right)}{\left[25 + 24\cos\left(\theta - \frac{\pi}{3}\right)\right]^2} d\theta = 0.$$

(12)

Next, we use Maple to verify the correctness of Eq. (12).

>evalf(int((159+18*cos(theta)+32*cos(theta-2*Pi/3)+360*cos(theta-Pi/3)+240*cos(2*theta-2*Pi/3))/(25+24*cos(theta-Pi/3))^2,theta=0..2*Pi),18);

0.

On the other hand, if $r = 6, s = -2, \varphi = -\pi/6,$ $m = 1, k = 1, a_0 = 3,$ and $a_1 = 4$ in Theorem 1, then using Eq. (4) yields

$$\int_0^{2\pi} \frac{864 + 36\sqrt{3} + 108\cos\theta + 12\cos\left(\theta + \frac{\pi}{3}\right) + 576\cos\left(\theta + \frac{\pi}{6}\right) + 96\cos\left(2\theta + \frac{\pi}{3}\right)}{\left[40 + 24\cos\left(\theta + \frac{\pi}{6}\right)\right]^2} d\theta$$

$$= \frac{4\pi}{3}.$$

(13)

We also use Maple to verify the correctness of Eq. (13).

>evalf(int((864+36*sqrt(3)+108*cos(theta)+12*cos(theta+Pi/3)+576*cos(theta+Pi/6)+96*cos(2*theta+Pi/3))/(40+24*cos(theta+Pi/6))^2,theta=0..2*Pi),18);

4.18879020478639098

>evalf(4*Pi/3,18);

4.18879020478639098

3.2. Example

In Theorem 2, let $r = 2, s = 7, \varphi = \pi/4, m = 1$, $k = 0, a_0 = 6,$ and $a_1 = 3$, by Eq. (10) we have

$$\int_0^{2\pi} \frac{12\sin\theta - 42\sin\left(\theta - \frac{\pi}{4}\right) - 42\sin\left(2\theta - \frac{\pi}{4}\right)}{53 - 28\cos\left(\theta - \frac{\pi}{4}\right)} d\theta = 0.$$

(14)

Using Maple to verify the correctness of Eq. (14) as follows:

>evalf(int((12*sin(theta)-42*sin(theta-Pi/4)-42*sin(2*theta-Pi/4))/(53-28*cos(theta-Pi/4)),theta=0..2*Pi),18);

0.

In addition, if $r = -8, s = -4, \varphi = -\pi/3,$ $m = 1, k = 0, a_0 = 7,$ and $a_1 = 2$ in Theorem 2, then using Eq. (11) yields

$$\int_0^{2\pi} \frac{128\sin\theta + 8\sin\left(\theta + \frac{\pi}{3}\right) - 64\sin\left(2\theta + \frac{\pi}{3}\right)}{80 - 64\cos\left(\theta + \frac{\pi}{3}\right)} d\theta$$

$$= -\sqrt{3}\pi.$$

(15)

The correctness of Eq. (15) can be verified by Maple.

>evalf(int((128*sin(theta)+8*sin(theta+Pi/3)-64*sin(2*theta+Pi/3))/(80-64*cos(theta+Pi/3)),theta=0..2*Pi),18);

-5.44139809270265354

>evalf(-sqrt(3)*Pi,18);

-5.44139809270265354

4. Conclusion

In this paper, Cauchy theorem and Cauchy integral formula for derivatives are used to evaluate two types of definite integrals. In fact, the applications of these two methods are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In

addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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