A Study on the Integrals of Rational Functions with Maple

Chii-Huei Yu
Department of Information Technology, Nan Jeon University of Science and Technology, Tainan City, Taiwan

Abstract*:*

This article uses the mathematical software Maple as an auxiliary tool to study the integral problems of rational functions. We can use binomial series and integration term by term theorem to find the infinite series form of the indefinite integral of some type of rational function, and hence greatly reducing the difficulty of evaluate its definite integral. On the other hand, we propose two rational functions to find their definite integrals, and the answers of these definite integrals are presented in infinite series forms. The research method adopted in this paper is to find out the answers, and then uses Maple to verify our results. This kind of research method not only allows us to find the calculation errors, but also can help us to amend the original thinking direction because we can verify the correctness of our theory from the consistency of manual and Maple calculations. Therefore, Maple can give us the inspiration of the problem and let us find the answers of the problems.

Keywords

Maple, indefinite integral, rational functions, binomial series, integration term by term theorem, infinite series form

1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics courses, there are many methods to solve the integral problems, for example, change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. On the other hand, Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve the integral problems. Moreover, Yu [4-27], Yu and Chen [28], and Yu and Sheu [29-31] used complex power series method, Parseval’s theorem, area mean value theorem, and generalized Cauchy integral formula to evaluate some types of integrals. This article studies the following indefinite integral of some type of rational function, which is not easy to obtain its answer using the methods mentioned above.

$$f(x)=\frac{\displaystyle\sum_{p=0}^{m}\frac{m!}{p!(m-p)!}b^p\cos p\theta\cdot x^p}{(1+2b\cos\theta\cdot x+b^2x^2)^m}, \qquad (1)$$

where θ, b, x are real numbers, $b \neq 0$, $x \neq \pm 1/b$, and m is a positive integer. Using binomial series and integration term by term theorem, the infinite series form of this type of integral can be determined, that is the major result of this article: Theorem A. Therefore, the difficulty of solving this problem can be greatly reduced. In addition, we propose two examples of rational functions, and actually evaluate their indefinite integrals and calculate some definite integrals. The research method used in this paper is to go through the process of obtaining the answers, and then use Maple to verify the answers. This approach not only let us find the calculation errors, but also help us revise the direction of the original thinking, because the consistency of the results obtained from manual and Maple calculations can verify the correctness of our theory.

2. Preliminaries and Results

First, we introduce some notations, formulas, and theorems used in this article.

2.1. Notations:

2.1.1. Suppose that $z=\alpha+i\beta$, where α,β are real numbers, and $i=\sqrt{-1}$. α, the real part of z, is denoted by $\operatorname{Re}(z)$; β the imaginary part of z, is denoted by $\operatorname{Im}(z)$.

2.1.2. Assume that a is a real number, and k is a positive integer. Define $(a)_k=a(a-1)\cdots(a-k+1)$, and $(a)_0=1$.

2.2. Formulas:

2.2.1. Euler's formula:

$e^{iy}=\cos y+i\sin y$, where y is a real number.

2.2.2. DeMoivre's formula:

$(\cos x+i\sin x)^n=\cos nx+i\sin nx$, where n is an integer, and x is a real number.

2.2.3. Binomial series:

If a is a real number, z is a complex number, and $|z|<1$, then $(1+z)^a=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k$..

2.3. Theorems:

2.3.1. Binomial theorem:

Suppose that w,z are complex numbers, and m is a non-negative integer, then

$$(w+z)^m=\sum_{p=0}^{m}\frac{m!}{p!(m-p)!}w^{m-p}z^p.$$

2.3.2. Integration term by term theorem:([32, p269])

Let $\{g_n\}_{n=0}^{\infty}$ be a sequence of Lebesgue integrable functions defined on I. If $\sum_{n=0}^{\infty}\int_I|g_n|$ is convergent, then $\int_I\sum_{n=0}^{\infty}g_n=\sum_{n=0}^{\infty}\int_I g_n$.

In the following, we obtain the major result in this paper, the infinite series form of indefinite integral of some type of rational function.

Theorem A *Suppose that θ,b,x are real numbers, $b\neq 0$, $x\neq\pm 1/b$, m is a positive integer, and let*

$$f(x)=\frac{\sum_{p=0}^{m}\frac{m!}{p!(m-p)!}b^p\cos p\theta\cdot x^p}{(1+2b\cos\theta\cdot x+b^2x^2)^m}.$$

Case 1. If $|x|<\frac{1}{|b|}$, then

$$\int f(x)dx=\sum_{k=0}^{\infty}\frac{(-m)_k}{k!(k+1)}b^k\cos k\theta\cdot x^{k+1}+C. \qquad (2)$$

Case 2. If $|x|>\frac{1}{|b|}$, then

$$\int f(x)dx$$

$$=\frac{-1}{b^m}\sum_{\substack{k=0\\k\neq 1-m}}^{\infty}\frac{(-m)_k}{k!(k+m-1)}\left(\frac{1}{b}\right)^k\cos(k+m)\theta\cdot\frac{1}{x^{k+m-1}}$$

$$+\delta(m)\cdot\frac{\cos\theta}{b}\cdot\ln|x|+C, \qquad (3)$$

where $\delta(m)=\begin{cases}1 & m=1\\ 0 & m\neq 1\end{cases}$.

Proof Since $f(x)$

$$=\frac{\sum_{p=0}^{m}\frac{m!}{p!(m-p)!}b^p\cos p\theta\cdot x^p}{(1+2b\cos\theta\cdot x+b^2x^2)^m}$$

$$=\frac{\operatorname{Re}[(1+be^{-i\theta}x)^m]}{[(1+be^{i\theta}x)\cdot(1+be^{-i\theta}x)]^m}$$

$$=\operatorname{Re}\left[\frac{(1+be^{-i\theta}x)^m}{(1+be^{i\theta}x)^m(1+be^{-i\theta}x)^m}\right]$$

$$=\operatorname{Re}\left[\frac{1}{(1+be^{i\theta}x)^m}\right]. \qquad (4)$$

Case 1. If $|x|<\frac{1}{|b|}$, then

$$f(x)$$

$$=\operatorname{Re}\left[\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}(be^{i\theta}x)^k\right]$$

(by Eq. (4) and binomial series)

$$=\operatorname{Re}\left[\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}b^ke^{ik\theta}x^k\right]$$

(by DeMoivre's formula)

$$=\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}b^k\cos k\theta\cdot x^k.$$

(by Euler's formula)

Hence, by integration term by term theorem, we obtain Eq. (2).

Case 2. If $|x|>\frac{1}{|b|}$, then

$$f(x)$$

$$= \mathrm{Re}\left[\frac{1}{(be^{i\theta}x)^m}\cdot\frac{1}{\left(1+\frac{1}{b}e^{-i\theta}\cdot\frac{1}{x}\right)^m}\right]$$

$$= \mathrm{Re}\left[\frac{1}{b^m e^{im\theta}x^m}\cdot\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}\left(\frac{1}{b}e^{-i\theta}\cdot\frac{1}{x}\right)^k\right]$$

(by DeMoivre's formula and binomial series)

$$= \frac{1}{b^m}\mathrm{Re}\left[\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}\left(\frac{1}{b}\right)^k e^{-i(k+m)\theta}\cdot\frac{1}{x^{k+m}}\right]$$

(by DeMoivre's formula)

$$= \frac{1}{b^m}\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}\left(\frac{1}{b}\right)^k \cos(k+m)\theta\cdot\frac{1}{x^{k+m}}.$$

(by Euler's formula)

Also, by integration term by term theorem, we obtain Eq. (3). q.e.d.

3. Examples

In the following, for the integral problem in this paper, we provide two examples and use Theorem A to determine their infinite series forms. Additionally, Maple is used to calculate the approximations of some definite integrals and their infinite series forms to verify our results.

Example 3.1. In Theorem A, let $\theta=\pi/3, b=-2$, $m=5$, and let the rational function

$$f(x)=\frac{1-5x-20x^2+80x^3-40x^4-16x^5}{(1-2x+4x^2)^5}. \quad (5)$$

Case 1. If $|x|<\frac{1}{2}$, then using Eq. (2) yields the indefinite integral

$$\int f(x)dx=\sum_{k=0}^{\infty}\frac{[5]_k 2^k}{k!(k+1)}\cos\frac{k\pi}{3}\cdot x^{k+1}+C. \quad (6)$$

Thus, we obtain the following definite integral

$$\int_{-1/8}^{1/4}\frac{1-5x-20x^2+80x^3-40x^4-16x^5}{(1-2x+4x^2)^5}dx$$

$$=\sum_{k=0}^{\infty}\frac{[5]_k 2^k}{k!(k+1)}\cos\frac{k\pi}{3}\left[\left(\frac{1}{4}\right)^{k+1}-\left(-\frac{1}{8}\right)^{k+1}\right]. \quad (7)$$

Next, we use Maple to verify the correctness of Eq. (7).

>evalf(int((1-5*x-20*x^2+80*x^3-40*x^4-16*x^5) /(1-2*x+4*x^2)^5,x=-1/8..1/4),20);

0.12814105233930306817

>evalf(sum(product(5+i,i=0..k-1)*2^k/(k!*(k+1))* cos(k*Pi/3)*((1/4)^(k+1)-(-1/8)^(k+1)),k=0..infinity), 20);

0.12814105233930306817

Case 2. If $|x|>\frac{1}{2}$, then by Eq. (3) we have

$$\int f(x)dx=\frac{1}{32}\cdot\sum_{k=0}^{\infty}\frac{[5]_k}{k!(k+4)}\left(\frac{1}{2}\right)^k\cos\frac{(k+5)\pi}{3}\cdot\frac{1}{x^{k+4}}+C. \quad (8)$$

Hence,

$$\int_2^7\frac{1-5x-20x^2+80x^3-40x^4-16x^5}{(1-2x+4x^2)^5}dx$$

$$=\frac{1}{32}\cdot\sum_{k=0}^{\infty}\frac{[5]_k}{k!(k+4)}\left(\frac{1}{2}\right)^k\cos\frac{(k+5)\pi}{3}\left(\frac{1}{7^{k+4}}-\frac{1}{2^{k+4}}\right). \quad (9)$$

Also, using Maple to verify the correctness of Eq. (9).

>evalf(int((1-5*x-20*x^2+80*x^3-40*x^4-16*x^5)/ (1-2*x+4*x^2)^5,x=2..7),20);

-0.00073483203676571326014

>evalf(1/32*sum(product(5+i,i=0..k-1)/(k!*(k+4))* (1/2)^k*cos((k+5)*Pi/3)*(1/7^(k+4)- 1/2^(k+4)),k=0.. infinity),20);

-0.00073483203676571326014

Example 3.2. Let $\theta=\cos^{-1}\frac{1}{2}, b=4$, and $m=1$ in Theorem A, and let

$$g(x)=\frac{1+2x}{1+4x+16x^2}. \quad (10)$$

Case 1. If $|x|<\frac{1}{4}$, then using Eq. (2) yields

$$\int g(x)dx=\sum_{k=0}^{\infty}\frac{(-4)^k}{k+1}\cos\left(k\cos^{-1}\frac{1}{2}\right)\cdot x^{k+1}+C. \quad (11)$$

Therefore,

$$\int_{-1/8}^{1/16}\frac{1+2x}{1+4x+16x^2}dx$$

$$= \sum_{k=0}^{\infty} \frac{(-4)^k}{k+1} \cos\left(k \cos^{-1} \frac{1}{2}\right) \left[\left(\frac{1}{16}\right)^{k+1} - \left(\frac{-1}{8}\right)^{k+1} \right]. \tag{12}$$

We also employ Maple to verify the correctness of Eq. (12).

>evalf(int((1+2*x)/(1+4*x+16*x^2),x=-1/8..1/16),20);

0.18950184761257350693

>evalf(sum((-4)^k/(k+1)*cos(k*arccos(1/2))*((1/16)^(k+1)-(-1/8)^(k+1)),k=0..infinity),20);

0.18950184761257350692

Case 2. If $|x| > \frac{1}{4}$, then by Eq. (3) we obtain

$$\int g(x)dx$$

$$= \frac{-1}{4} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{-1}{4}\right)^k \cos\left((k+1)\cos^{-1} \frac{1}{2}\right) \frac{1}{x^k} + \frac{1}{8} \ln|x| + C. \tag{13}$$

Thus,

$$\int_2^5 \frac{1+2x}{1+4x+16x^2} dx$$

$$= \frac{-1}{4} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{-1}{4}\right)^k \cos\left((k+1)\cos^{-1} \frac{1}{2}\right) \left(\frac{1}{5^k} - \frac{1}{2^k}\right) + \frac{1}{8} \ln \frac{5}{2}. \tag{14}$$

Using Maple to verify the correctness of Eq. (14).

>evalf(int((1+2*x)/(1+4*x+16*x^2),x=2..5),20);

0.12235289808692021233

>evalf(-1/4*sum(1/k*(-1/4)^k*cos((k+1)*arccos(1/2))*(1/5^k-1/2^k),k=1..infinity)+1/8*ln(5/2),20);

0.12235289808692021236

4. Conclusion

In this study, we mainly use binomial series and integration term by term theorem to study the integral problem of some type of rational function. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers.

References

[1] A. A. Adams, H. Gottliebsen, S. A. Linton, and U. Martin, "Automated theorem proving in support of computer algebra: symbolic definite integration as a case study," *Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation*, Canada, pp. 253-260, 1999.

[2] M. A. Nyblom, "On the evaluation of a definite integral involving nested square root functions," *Rocky Mountain Journal of Mathematics*, Vol. 37, No. 4, pp. 1301-1304, 2007.

[3] C. Oster, "Limit of a definite integral," *SIAM Review*, Vol. 33, No. 1, pp. 115-116, 1991.

[4] C. -H. Yu, "Solving some definite integrals using Parseval's theorem," *American Journal of Numerical Analysis*, Vol. 2, No. 2, pp. 60-64, 2014.

[5] C. -H. Yu, "Some types of integral problems," *American Journal of Systems and Software*, Vol. 2, No. 1, pp. 22-26, 2014.

[6] C. -H. Yu, "Using Maple to study the double integral problems," *Applied and Computational Mathematics*, Vol. 2, No. 2, pp. 28-31, 2013.

[7] C. -H. Yu, " A study on double integrals, " *International Journal of Research in Information Technology*, Vol. 1, Issue. 8, pp. 24-31, 2013.

[8] C. -H. Yu, "Application of Parseval's theorem on evaluating some definite integrals," *Turkish Journal of Analysis and Number Theory*, Vol. 2, No. 1, pp. 1-5, 2014.

[9] C. -H. Yu, "Evaluation of two types of integrals using Maple, " *Universal Journal of Applied Science*, Vol. 2, No. 2, pp. 39-46, 2014.

[10] C. -H. Yu, "Studying three types of integrals with Maple, " *American Journal of Computing Research Repository*, Vol. 2, No. 1, pp. 19-21, 2014.

[11] C. -H. Yu, "The application of Parseval's theorem to integral problems, " *Applied Mathematics and Physics*, Vol. 2, No. 1, pp. 4-9, 2014.

[12] C. -H. Yu, "A study of some integral problems using Maple, " *Mathematics and Statistics*, Vol. 2, No. 1, pp. 1-5, 2014.

[13] C. -H. Yu, "Solving some definite integrals by using Maple, " *World Journal of Computer Application and Technology*, Vol. 2, No. 3, pp. 61-65, 2014.

[14] C. -H. Yu, "Using Maple to study two types of integrals, " *International Journal of Research in Computer Applications and Robotics*, Vol. 1, Issue. 4, pp. 14-22, 2013.

[15] C. -H. Yu, "Solving some integrals with Maple," *International Journal of Research in Aeronautical and Mechanical Engineering*, Vol. 1, Issue. 3, pp. 29-35, 2013.

[16] C. -H. Yu, "A study on integral problems by using Maple, " *International Journal of Advanced Research in Computer Science and Software Engineering*, Vol. 3, Issue. 7, pp. 41-46, 2013.

[17] C. -H. Yu, "Evaluating some integrals with Maple, " *International Journal of Computer Science and Mobile Computing*, Vol. 2, Issue. 7, pp. 66-71, 2013.

[18] C. -H. Yu, "Application of Maple on evaluation of definite integrals, " *Applied Mechanics and Materials*, Vols. 479-480 (2014), pp. 823-827, 2013.

[19] C. -H. Yu, "Application of Maple on the integral problems, " *Applied Mechanics and Materials*, Vols. 479-480 (2014), pp. 849-854, 2013.

[20] C. -H. Yu, "Using Maple to study the integrals of trigonometric functions, " *Proceedings of the 6th IEEE/International Conference on Advanced Infocomm Technology*, Taiwan, No. 00294, 2013.

[21] C. -H. Yu, "A study of the integrals of trigonometric functions with Maple," *Proceedings of the Institute of Industrial Engineers Asian Conference 2013*, Taiwan, Springer, Vol. 1, pp. 603-610, 2013.

[22] C. -H. Yu, "Application of Maple on the integral problem of some type of rational functions," (in Chinese) *Proceedings of the Annual Meeting and Academic Conference for Association of IE*, Taiwan, D357-D362, 2012.

[23] C. -H. Yu, "Application of Maple on some integral problems," (in Chinese) *Proceedings of the International Conference on Safety & Security Management and Engineering Technology 2012*, Taiwan, pp. 290-294, 2012.

[24] C. -H. Yu, "Application of Maple on some type of integral problem," (in Chinese) *Proceedings of the Ubiquitous-Home Conference 2012*, Taiwan, pp.206-210, 2012.

[25] C. -H. Yu, "Application of Maple on evaluating the closed forms of two types of integrals," (in Chinese) *Proceedings of the 17th Mobile Computing Workshop*, Taiwan, ID16, 2012.

[26] C. -H. Yu, "Application of Maple: taking two special integral problems as examples," (in Chinese) *Proceedings of the 8th International Conference on Knowledge Community*, Taiwan, pp.803-811, 2012.

[27] C. -H. Yu, "Evaluating some types of definite integrals," *American Journal of Software Engineering*, Vol. 2, Issue. 1, pp. 13-15, 2014.

[28] C. -H. Yu and B. -H. Chen, "Solving some types of integrals using Maple," *Universal Journal of Computational Mathematics*, Vol. 2, No. 3, pp. 39-47, 2014.

[29] C. -H. Yu and S. -D. Sheu, "Using area mean value theorem to solve some double integrals," *Turkish Journal of Analysis and Number Theory*, Vol. 2, No. 3, pp. 75-79, 2014.

[30] C. -H. Yu and S. -D. Sheu, "Infinite series forms of double integrals," *International Journal of Data Envelopment Analysis and *Operations Research**, Vol. 1, No. 2, pp. 16-20, 2014.

[31] C. -H. Yu and S. -D. Sheu, "Evaluation of triple integrals," *American Journal of Systems and Software*, Vol. 2, No. 4, pp. 85-88, 2014.

[32] T. M. Apostol, *Mathematical Analysis*, 2nd ed., Massachusetts: Addison-Wesley, 1975.