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Complex Analysis Method for Solving Double Integral Problems

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Abstract:

In this article, the mathematical software Maple is used as an auxiliary tool to study two types of double integrals. The infinite series forms of these two types of double integrals can be determined by complex analysis method. In addition, two examples of double integrals are proposed and we actually find their infinite series forms. The research method adopted in this paper is to find out the answers by manual calculation, and then uses Maple to verify the answers. This research method not only allows us to find the calculation errors, but also can help us to amend the original thinking direction because we can verify the correctness of our theory from the consistency of manual and Maple calculations.

Keywords

double integrals, infinite series forms, complex analysis method, Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, we can refer to [1-5].

In calculus and engineering mathematics, the area of the surface, the volume under the surface, and the centroid position of the sheet are involved in double integrals, so the numerical calculation of double integrals is an important issue. Adams et al. [6], Nyblom [7], and Oster [8] provided some methods to solve the integral problems. Yu [9-32], Yu and Chen [33], Yu and Sheu [34-36] used complex power series method, integration term by term theorem, differentiation with respect to a parameter, Parseval's theorem, area mean value theorem, and generalized Cauchy integral formula to solve some types of integral problems. This paper studies the following two types of double integrals

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\cos(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta\,, \quad (1)$$

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\sin(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta\,, \quad (2)$$

where $r_1, r_2, \theta_1, \theta_2$ are real numbers, and m, p are positive integers. Using binomial series and integration term by term theorem, the infinite series form of these two types of double integrals can be determined, these are the major results of this article: Theorems 1 and 2. Thus, the difficulty of solving the double integral problems can be greatly reduced. On the other hand, we propose some examples to do calculation practically. The research method used in this paper is to go through the process of obtaining the answers, and then use Maple to verify the answers. This approach not only let us find the calculation errors, but also help us revise the direction of the original thinking, because the consistency of the results obtained from manual and Maple calculations can verify the correctness of our theory.

2. Preliminaries and Main Results

First, we introduce some notations, formulas, and theorems used in this article.

2.1. Notations:

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2.1.1. Suppose that $z=\alpha+i\beta$, where α,β are real numbers, and $i=\sqrt{-1}$. α, the real part of z, is denoted as $\operatorname{Re}(z)$; β the imaginary part of z, is denoted as $\operatorname{Im}(z)$, and the conjugate complex number of z, is denoted as $\bar{z}$.

2.1.2. Assume that a is a real number, and k is a positive integer. Define $(a)_k = a(a-1)\cdots(a-k+1)$, and $(a)_0 = 1$.

2.2. Formulas:

2.2.1. Euler's formula:

$e^{iy} = \cos y + i\sin y$, where y is a real number.

2.2.2. DeMoivre's formula:

$(\cos x + i\sin x)^n = \cos nx + i\sin nx$, where n is an integer, and x is a real number.

2.2.3. Binomial series:

If a is a real number, z is a complex number, and $|z|<1$, then $(1+z)^a = \sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k$..

2.3. Theorems:

2.3.1. Binomial theorem:

Assume that u,v are complex numbers, and m is a non-negative integer, then

$$(u+v)^m = \sum_{n=0}^{m}\frac{m!}{n!(m-n)!}u^{m-n}v^n.$$

2.3.2. Integration term by term theorem:([37, p269])

Let $\{g_n\}_{n=0}^{\infty}$ be a sequence of Lebesgue integrable functions defined on I. If $\sum_{n=0}^{\infty}\int_I |g_n|$ is convergent, then $\int_I \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty}\int_I g_n$.

To obtain the main results in this paper, we need the following lemma:

Lemma *Assume that* r,θ *are real numbers,* m,p *are positive integers, and let* $z=re^{i\theta}$, *then*

$$\operatorname{Re}\left(\frac{z^p}{(1+z)^m}\right) = \frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\cos(p-n)\theta}{(1+2r\cos\theta+r^2)^m}, \tag{3}$$

and

$$\operatorname{Im}\left(\frac{z^p}{(1+z)^m}\right) = \frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\sin(p-n)\theta}{(1+2r\cos\theta+r^2)^m}. \tag{4}$$

Proof $\operatorname{Re}\left(\frac{z^p}{(1+z)^m}\right)$

$$= \operatorname{Re}\left(\frac{z^p(1+\bar{z})^m}{(1+z)^m(1+\bar{z})^m}\right)$$

$$= \operatorname{Re}\left(\frac{z^p\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}\bar{z}^n}{(1+z+\bar{z}+|z|^2)^m}\right)$$

(by binomial theorem)

$$= \operatorname{Re}\left(\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}e^{i(p-n)\theta}}{(1+2r\cos\theta+r^2)^m}\right)$$

$$= \frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\cos(p-n)\theta}{(1+2r\cos\theta+r^2)^m}.$$

Similarly,

$$\operatorname{Im}\left(\frac{\bar{z}^p}{(1+z)^m}\right) = \frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\sin(p-n)\theta}{(1+2r\cos\theta+r^2)^m}.$$

q.e.d.

In the following, the infinite series form of double integral (1) can be determined.

Theorem 1 *Suppose that* $r_1,r_2,\theta_1,\theta_2$ *are real numbers, and* m,p *are positive integers.*

Case 1. If $|r_1|<1, |r_2|<1$, *then*

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\cos(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta$$

$$= \sum_{k=0}^{\infty}\frac{(-m)_k[r_2^{k+p+1}-r_1^{k+p+1}][\sin(p+k)\theta_2-\sin(p+k)\theta_1]}{k!(k+p+1)(p+k)}. \tag{5}$$

Case 2. If $|r_1|>1, |r_2|>1$, *and* $p-m+1<0$, *then*

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\cos(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta$$

$$= \sum_{k=0}^{\infty}\frac{(-m)_k[r_2^{p-m-k+1}-r_1^{p-m-k+1}][\sin(p-m-k)\theta_2-\sin(p-m-k)\theta_1]}{k!(p-m-k+1)(p-m-k)}. \tag{6}$$

Proof Let $z=re^{i\theta}$.

Case 1. If $|r_1|<1, |r_2|<1$. Since

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$$\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\cos(p-n)\theta}{(1+2r\cos\theta+r^2)^m}$$

$$=\mathrm{Re}\left(\frac{z^p}{(1+z)^m}\right) \text{ (by Eq. (3))}$$

$$=\mathrm{Re}\left(z^p\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}z^k\right)$$

(since $|z|<1$, we can use binomial series)

$$=\mathrm{Re}\left(\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{k+p}e^{i(p+k)\theta}\right)$$

(by DeMoivre's formula)

$$=\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{k+p}\cos(p+k)\theta\ . \tag{7}$$

(by Euler's formula)

If follows that the double integral

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\cos(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta$$

$$=\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{k+p}\cos(p+k)\theta drd\theta$$

(by Eq. (7))

$$=\int_{\theta_1}^{\theta_2}\sum_{k=0}^{\infty}\frac{(-m)_k[r_2^{k+p+1}-r_1^{k+p+1}]}{k!(k+p+1)}\cos(p+k)\theta drd\theta$$

(by integration term by term theorem)

$$=\sum_{k=0}^{\infty}\frac{(-m)_k[r_2^{k+p+1}-r_1^{k+p+1}][\sin(p+k)\theta_2-\sin(p+k)\theta_1]}{k!(k+p+1)(p+k)}.$$

(again by integration term by term theorem)

Case 2. If $|r_1|>1, |r_2|>1$, and $p-m+1<0$. Since

$$\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\cos(p-n)\theta}{(1+2r\cos\theta+r^2)^m}$$

$$=\mathrm{Re}\left(\frac{z^p}{(1+z)^m}\right) \text{ (by Eq. (3))}$$

$$=\mathrm{Re}\left(\frac{z^{p-m}}{(1+z^{-1})^m}\right)$$

$$=\mathrm{Re}\left(z^{p-m}\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}z^{-k}\right)$$

(since $|z^{-1}|>1$, we can use binomial series)

$$=\mathrm{Re}\left(\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{p-m-k}e^{i(p-m-k)\theta}\right)$$

(by DeMoivre's formula)

$$=\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{p-m-k}\cos(p-m-k)\theta\ . \tag{8}$$

Hence, the double integral

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\cos(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta$$

$$=\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{p-m-k}\cos(p-m-k)\theta drd\theta$$

$$=\sum_{k=0}^{\infty}\frac{(-m)_k[r_2^{p-m-k+1}-r_1^{p-m-k+1}][\sin(p-m-k)\theta_2-\sin(p-m-k)\theta_1]}{k!(p-m-k+1)(p-m-k)}.$$

(by integration term by term theorem)

q.e.d.

Next, we obtain the infinite series form of double integral (2).

Theorem 2 *If the assumptions are the same as Theorem 1.*

Case 1. If $|r_1|<1, |r_2|<1$, then

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\sin(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta$$

$$=-\sum_{k=0}^{\infty}\frac{(-m)_k[r_2^{k+p+1}-r_1^{k+p+1}][\cos(p+k)\theta_2-\cos(p+k)\theta_1]}{k!(k+p+1)(p+k)}. \tag{9}$$

Case 2. If $|r_1|>1, |r_2|>1$, and $p-m+1<0$, then

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\sin(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta$$

$$=-\sum_{k=0}^{\infty}\frac{(-m)_k[r_2^{p-m-k+1}-r_1^{p-m-k+1}][\cos(p-m-k)\theta_2-\cos(p-m-k)\theta_1]}{k!(p-m-k+1)(p-m-k)}. \tag{10}$$

Proof Also, let $z=re^{i\theta}$.

Case 1. If $|r_1|<1, |r_2|<1$. Since

$$\frac{\displaystyle\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\sin(p-n)\theta}{(1+2r\cos\theta+r^2)^m}$$

$$=\operatorname{Im}\left(\frac{z^p}{(1+z)^m}\right)\text{ (by Eq. (4))}$$

$$=\operatorname{Im}\left(z^p\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}z^k\right)$$

$$=\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{k+p}\sin(p+k)\theta\ . \tag{11}$$

Thus, using integration term by term theorem yields

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\displaystyle\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\sin(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta$$

$$=\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{k+p}\sin(p+k)\theta drd\theta$$

(by Eq. (11))

$$=-\sum_{k=0}^{\infty}\frac{(-m)_k[r_2^{k+p+1}-r_1^{k+p+1}][\cos(p+k)\theta_2-\cos(p+k)\theta_1]}{k!(k+p+1)(p+k)}.$$

Case 2. If $|r_1|>1,|r_2|>1$, and $p-m+1<0$. Since

$$\frac{\displaystyle\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\sin(p-n)\theta}{(1+2r\cos\theta+r^2)^m}$$

$$=\operatorname{Im}\left(\frac{z^{p-m}}{(1+z^{-1})^m}\right)$$

$$=\operatorname{Im}\left(\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{p-m-k}e^{i(p-m-k)\theta}\right)$$

$$=\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{p-m-k}\sin(p-m-k)\theta\ . \tag{12}$$

Therefore, also by integration term by term theorem, we have

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{\displaystyle\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}r^{n+p}\sin(p-n)\theta}{(1+2r\cos\theta+r^2)^m}drd\theta$$

$$=\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\sum_{k=0}^{\infty}\frac{(-m)_k}{k!}r^{p-m-k}\sin(p-m-k)\theta drd\theta$$

$$=-\sum_{k=0}^{\infty}\frac{(-m)_k[r_2^{p-m-k+1}-r_1^{p-m-k+1}][\cos(p-m-k)\theta_2-\cos(p-m-k)\theta_1]}{k!(p-m-k+1)(p-m-k)}.$$

q.e.d.

3. Examples

For the double integral problem discussed in this paper, we will provide some examples and use Theorems 1 and 2 to determine their infinite series forms. In addition, Maple is used to calculate the approximations of some double integrals and their infinite series forms to verify our answers.

Example 3.1. In Theorem 1, if $m=3, p=4$, $r_1=1/3, r_2=1/2$, $\theta_1=\pi/6, \theta_2=\pi/3$, then by Eq. (5) we have

$$\int_{\pi/6}^{\pi/3}\int_{1/3}^{1/2}\frac{\displaystyle\sum_{n=0}^{3}\frac{3!}{n!(3-n)!}r^{n+4}\cos(4-n)\theta}{(1+2r\cos\theta+r^2)^3}drd\theta$$

$$=\sum_{k=0}^{\infty}\frac{(-3)_k[(1/2)^{k+5}-(1/3)^{k+5}][\sin(k+4)\pi/3-\sin(k+4)\pi/6]}{k!(k+5)(k+4)}. \tag{13}$$

We employ Maple to verify the correctness of Eq. (13) as follows:

>evalf(Doubleint(sum(3!/(n!*(3-n)!)*r^(n+4)*cos((4-n)*theta),n=0..3)/(1+2*r*cos(theta)+r^2)^3,r=1/3..1/2,theta=Pi/6..Pi/3),14);

−0.00083653550898154

>evalf(sum(product(-3-j,j=0..(k-1))*((1/2)^(k+5)-(1/3)^(k+5))*(sin((k+4)*Pi/3)-sin((k+4)*Pi/6))/(k!*(k+5)*(k+4)),k=0..infinity),14);

−0.00083653550898155

On the other hand, if $m=5, p=2$, $r_1=3, r_2=6$, $\theta_1=\pi/2, \theta_2=\pi$ in Theorem 1, then using Eq. (6) yields

$$\int_{\pi/2}^{\pi}\int_{3}^{6}\frac{\displaystyle\sum_{n=0}^{5}\frac{5!}{n!(5-n)!}r^{n+2}\cos(2-n)\theta}{(1+2r\cos\theta+r^2)^5}drd\theta$$

$$=\sum_{k=0}^{\infty}\frac{(-5)_k(6^{-2-k}-3^{-2-k})[\sin(-3-k)\pi-\sin(-3-k)\pi/2]}{k!(-2-k)(-3-k)}. \tag{14}$$

Using Maple to verify the correctness of Eq. (14) as follows:

>evalf(Doubleint(sum(5!/(n!*(5-n)!)*r^(n+2)*cos((2-n)*theta),n=0..5)/(1+2*r*cos(theta)+r^2)^5,r=3..6,theta=Pi/2..Pi),14);

0.0070816832171841

>evalf(sum(product(-5-j,j=0..(k-1))*(6^(-2-k)-3^(-2-

k))*(sin((-3-k)*Pi)-sin((-3-k)*Pi/2))/(k!*(-2-k)*(-3-k)),k=0..infinity),14);

0.0070816832171836

Example 3.2. In Theorem 2, let $m=6, p=7$, $r_1=1/5, r_2=1/3$, $\theta_1=\pi/4, \theta_2=\pi/2$, then using Eq. (9) yields

$$\int_{\pi/4}^{\pi/2}\int_{1/5}^{1/3}\frac{\sum_{n=0}^{6}\frac{6!}{n!(6-n)!}r^{n+7}\sin(7-n)\theta}{(1+2r\cos\theta+r^2)^6}drd\theta$$

$$=-\sum_{k=0}^{\infty}\frac{(-6)_k[(1/3)^{k+8}-(1/5)^{k+8}][\cos(k+7)\pi/2-\cos(k+7)\pi/4]}{k!(k+8)(k+7)}. \quad (15)$$

Using Maple to verify the correctness of Eq. (15) .

```
>evalf(Doubleint(sum(6!/(n!*(6-n)!)*r^(n+7)*sin((7-n)*theta),n=0..6)/(1+2*r*cos(theta)+r^2)^6,r=1/5..1/3,theta=Pi/4..Pi/2),14);
```

0.0000020196179825389

```
>evalf(-sum(product(-6-j,j=0..(k-1))*((1/3)^(k+8)-(1/5)^(k+8))*(cos((k+7)*Pi/2)-cos((k+7)*Pi/4))/(k!*(k+8)*(k+7)),k=0..infinity),14);
```

0.0000020196179825390

In addition, if $m=3, p=1, r_1=2, r_2=9$, $\theta_1=\pi/3, \theta_2=\pi$ in Theorem 2, then by Eq. (10) we have

$$\int_{\pi/3}^{\pi}\int_{2}^{9}\frac{\sum_{n=0}^{3}\frac{3!}{n!(3-n)!}r^{n+1}\sin(1-n)\theta}{(1+2r\cos\theta+r^2)^3}drd\theta$$

$$=-\sum_{k=0}^{\infty}\frac{(-3)_k(9^{-1-k}-2^{-1-k})[\cos(-2-k)\pi-\cos(-2-k)\pi/3]}{k!(-1-k)(-2-k)}. \quad (16)$$

We also use Maple to verify the correctness of Eq. (16).

```
>evalf(Doubleint(sum(3!/(n!*(3-n)!)*r^(n+1)*sin((1-n)*theta),n=0..3)/(1+2*r*cos(theta)+r^2)^3,r=2..9,theta=Pi/3..Pi),14);
```

0.45123626373626

```
>evalf(-sum(product(-3-j,j=0..(k-1))*(9^(-1-k)-2^(-1-k))*(cos((-2-k)*Pi)-cos((-2-k)*Pi/3))/(k!*(-1-k)*(-2-k)),k=0..infinity),14);
```

0.45123626373626

4. Conclusion

In this paper, we use binomial series and integration term by term theorem to study two types of double integral problems. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the subject of the study to another mathematical problems in calculus and engineering mathematics and use Maple to verify our answers.

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