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Some Improper Integral Problems

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Abstract:

This paper takes advantage of the mathematical software Maple to study two types of improper integrals. The infinite series forms of these two types of improper integrals can be obtained by using differentiation with respect to a parameter and differentiation term by term theorem. In addition, two examples of improper integrals are proposed and we actually find their infinite series forms. The research method adopted in this article is to obtain the answers by manual calculation, and then uses Maple to verify the answers. This research method not only allows us to find the calculation errors, but also helps us to amend the original thinking direction because we can verify the correctness of our theory from the consistency of manual and Maple calculations.

Keywords

Maple, improper integrals, infinite series forms, differentiation with respect to a parameter, differentiation term by term theorem

1. Introduction

Computer Algebra System (CAS) has been widely used in mathematical and scientific research. Through the computer's fast computing and beautiful affinity of the graphical interface, mathematical and scientific research increase the infinite imagination. The mathematical software Maple can be said to be the leader in the field of CAS, which occupies a pivotal position in the mathematical operation system. Maple The advantage of this software is that its instructions are simple and easy to learn, allowing people who engaged in mathematical and scientific research to save a lot of time of learning computer programming language, most of the spirit into the study of the problem. On the other hand, through Maple's numerical and symbolic operations, the thinking logic is transformed into a series of instructions, and the results of Maple's operations are used to correct the direction of previous inferences and reflections. Because this feedback is straightforward and constructive, it can enhance our understanding of the problem and the interest in research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, we can refer to [1-5].

In the course of advanced calculus or engineering mathematics, the study of improper integrals is an important issue. For example, the Gamma function and the Beta function and some other special functions are presented in the form of improper integrals. Therefore, the numerical calculation of improper integrals is important in physics, engineering, or other natural sciences. On the other hand, Adams et al. [6], Nyblom [7], and Oster [8] provided some methods to solve the integral problems. Yu [9-32], Yu and Chen [33], Yu and Sheu [34-36] used complex power series method, integration term by term theorem, differentiation with respect to a parameter, Parseval's theorem, area mean value theorem, and generalized Cauchy integral formula to solve some types of integral problems. This paper mainly studies the following two types of improper integrals:

$$\int_0^{\infty} x^{2m} e^{-ax^2-b/x^2} dx, \tag{1}$$

$$\int_0^{\infty} x^{2m} e^{-ax^2} \cos bx dx, \tag{2}$$

where $a>0, b>0$, and m is a non-negative integer. Using differentiation with a parameter and differentiation term by term theorem, we can obtain the infinite series forms of the two types of improper integrals, that is, the main results: Theorems 1 and 2. Therefore, the difficulty of solving the improper integral problems can be greatly reduced. In addition, we provide some examples to demonstrate the calculation practically. The research method adopted in this paper is to go through the process of obtaining the answers, and then use Maple to verify the answers. This approach not only let us find the calculation errors, but also help us revise the direction of the original thinking, because the consistency of the results obtained from manual and Maple calculations can verify the correctness of our theory.

2. Preliminaries and Results

At first, we introduce some notations, formulas, and theorems used in this paper.

2.1. Notations:

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Suppose that t is a real number, and m is a positive integer. Define $(t)_m = t(t-1)\cdots(t-m+1)$, and $(t)_0 = 1$.

2.2. Formulas: ([38, p452, Formula 652 & 664])

2.2.1. $\int_0^\infty e^{-ax^2-b/x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\cdot e^{-2\sqrt{ab}}$, where $a>0, b>0$.

2.2.2. $\int_0^\infty e^{-ax^2}\cos bx dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\cdot e^{-b^2/(4a)}$, where $a>0, b>0$.

2.3. Theorems:

2.3.1. Differentiation with respect to a parameter ([39, p405]):

Suppose that I_1, I_2 are real intervals, and assume that two variables function $f(x,y)$ and its first order partial derivative with respect to y, $f_y(x,y)$ are all defined on $I_1 \times I_2$. If the following two conditions are satisfied: (i) For all $y \in I_2$, Lebesgue integrals $\int_{I_1} f(x,y)dx$ and $\int_{I_1} f_y(x,y)dx$ are all exist, (ii) $f_y(x,y)$ is a continuous function defined on $I_1 \times I_2$ such that $\int_{I_1} f_y(x,y)dx$ is uniformly convergent on I_2. Then $F(y) = \int_{I_1} f(x,y)dx$ is differentiable on I_2, and its derivative $\frac{d}{dy}F(y) = \int_{I_1} f_y(x,y)dx$.

2.3.2. Differentiation term by term theorem ([37, p230]):

For all non-negative integers k, if the functions $g_k : (a,b) \to R$ satisfy the following three conditions: (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on the open interval (a,b), and (iii) $\sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$ is uniformly convergent on (a,b), then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b). Moreover, the derivative $\frac{d}{dx}\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$.

Next, we determine the infinite series form of improper integral (1).

Theorem 1 *Assume that $a>0, b>0$, and m is a non-negative integer, then*

$$\int_0^\infty x^{2m}e^{-ax^2-b/x^2}dx$$

$$= \frac{(-1)^m\sqrt{\pi}}{2}\cdot\sum_{k=0}^{\infty}\frac{(-2)^k((k-1)/2)_m}{k!}\cdot a^{[(k-1)/2]-m}\cdot b^{k/2}. \tag{3}$$

Proof Since $\int_0^\infty e^{-ax^2-b/x^2}dx$

$$= \frac{1}{2}\sqrt{\frac{\pi}{a}}\cdot e^{-2\sqrt{ab}} \quad \text{(by Formula 2.2.1)}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{a}}\cdot\sum_{k=0}^{\infty}\frac{1}{k!}(-2\sqrt{ab})^k$$

$$= \frac{\sqrt{\pi}}{2}\cdot\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}\cdot a^{(k-1)/2}\cdot b^{k/2}. \tag{4}$$

Using differentiation with respect to a parameter and differentiation term by term theorem, differentiating m times with respect to a on both sides of Eq. (4), the desired result holds. q.e.d.

In the following, we determine the infinite series form of improper integral (2).

Theorem 2 *If the assumptions are the same as Theorem 1, then*

$$\int_0^\infty x^{2m}e^{-ax^2}\cos bx dx$$

$$= \frac{(-1)^m\sqrt{\pi}}{2}\cdot\sum_{k=0}^{\infty}\frac{(-1/4)^k(-(2k+1)/2)_m}{k!}\cdot a^{[-(2k+1)/2]-m}\cdot b^{2k}. \tag{5}$$

Proof Since $\int_0^\infty e^{-ax^2}\cos bx dx$

$$= \frac{1}{2}\sqrt{\frac{\pi}{a}}\cdot e^{-b^2/(4a)} \quad \text{(by Formula 2.2.2)}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{a}}\cdot\sum_{k=0}^{\infty}\frac{1}{k!}[-b^2/(4a)]^k$$

$$= \frac{\sqrt{\pi}}{2}\cdot\sum_{k=0}^{\infty}\frac{(-1/4)^k}{k!}\cdot a^{-(2k+1)/2}\cdot b^{2k}. \tag{6}$$

Also, by differentiation with respect to a parameter and differentiation term by term theorem, differentiating m times with respect to a on both sides of Eq. (6), the desired result holds. q.e.d.

3. Examples

In the following, for the improper integral problems discussed in this article, some examples are provided and we use Theorems 1 and 2 to obtain their infinite series forms. On the other hand, Maple is used to calculate the approximations of some improper integrals and their infinite series forms to verify our answers.

Example 3.1. Using Theorem 1 yields

$$\int_0^{\infty} x^{20} e^{-3x^2-4/x^2} dx$$
$$= \frac{\sqrt{\pi}}{2} \cdot \sum_{k=0}^{\infty} \frac{(-2)^k ((k-1)/2)_{10}}{k!} \cdot 3^{[(k-21)/2]} \cdot 4^{k/2}. \quad (7)$$

We employ Maple to verify the correctness of Eq. (7) as follows:

```
>evalf(int(x^20*exp(-3*x^2-4/x^2),x=0..infinity),20);
1.6936129291376077076
>evalf(sqrt(Pi)/2*sum((-2)^k*product((k-1)/2-j,j=0..9)/k!*3^((k-21)/2)*4^(k/2),k=0..infinity),20);
1.6936129291376077076
```

Example 3.2. By Theorem 2, we have

$$\int_0^{\infty} x^{38} e^{-6x^2} \cos 10x dx$$
$$= -\frac{\sqrt{\pi}}{2} \cdot \sum_{k=0}^{\infty} \frac{(-1/4)^k (-(2k+1)/2)_{19}}{k!} \cdot 6^{[-(2k+39)/2]} \cdot 10^{2k}. \quad (8)$$

Using Maple to verify the correctness of Eq. (8) as follows:

```
>evalf(int(x^38*exp(-6*x^2)*cos(10*x),x=0..infinity),20);
0.53208107422709030716
>evalf(-sqrt(Pi)/2*sum((-1/4)^k*product((-2*k-1)/2-j,j=0..18)/k!*6^((-2*k-39)/2)*10^(2*k),k=0..infinity),20);
0.53208107422709030716
```

4. Conclusion

As mentioned, we use differentiation with respect to a parameter and differentiation term by term theorem to solve two types of improper integrals. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. Moreover, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the study subject to another mathematical problems and use Maple to verify our answers.

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