A Study on Fourier Series with Maple

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Abstract:

In this paper, we use the mathematical software Maple as a problem-solving tool to study the Fourier series expansion problem of absolutely integrable periodic functions. The way we study is to get the answers by manual calculation, and then use Maple to verify the answers. This kind of research not only allows us to find the calculation errors, but also can help us to amend the direction of the original thinking, because we can verify the correctness of the theory from the consistency of manual and Maple calculations.

Keywords:

Maple, Fourier series expansion, absolutely integrable , periodic functions

1. Introduction

This article uses the mathematical software Maple as an auxiliary tool in mathematics teaching and research. The main reason we choose Maple is that its instructions are easy to learn, even beginners are also very easy to use, so we can save a lot of time to learn computer programming language. On the other hand, through Maple's numerical and symbolic operations, the thinking logic is transformed into a series of instructions, and the results of Maple's operations are used to correct the direction of previous inferences and reflections. Because this feedback is straightforward and constructive, it can enhance our understanding of the problem and the interest in research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, we can refer to [1-5].

In this paper, some practical examples are proposed to study the Fourier series expansion method of the absolutely integrable periodic function. At the same time, with Maple's superior drawing function allows us to understand how the Fourier series pointwise converges to the original function. As for the introduction and application of the Fourier series theory can be found in [6-7].

2. Main Theory

First, we introduce some notations used in this article.

2.1. Notations:

2.1.1. Closed interval $[a,b]=\{x\in R|a\le x\le b\}$, and open interval $(a,b)=\{x\in R|a<x<b\}$.

2.1.2. Some instructions of Maple:

>r1:=sum(an*sin(n*x),n=1..p);

Define $r1=\sum_{n=1}^{p} a_n \sin nx$.

>sum(bn,n=1..infinity);

Calculating infinite series $\sum_{n=1}^{\infty} b_n$.

>int(f(x),x=a..b);

Find the definite integral of function $f(x)$ from $x=a$ to $x=b$.

>f:=piecewise( );

Define f is a piecewise function.

>plot(f(x),x=a..b);

Draw a graph of function $f(x)$ from $x=a$ to $x=b$.

>seq(an,n=1..k);

List the first k terms of sequence $\{a_n\}$.

Next, there are some definitions and theorems used in this paper.

2.2. Definitions:

2.2.1. If the function $f(x)$ satisfies $f(-x)=f(x)$, then it is called even function; if $f(-x)=-f(x)$, then it is an odd function.

2.2.2. The function $f(x)$ defined on the interval I is called absolutely integrable if it satisfies $\int_I |f(x)|dx<\infty$.

2.2.3. The function $f(x)$ is defined on the interval I, if it is differentable on each subinterval of I, then it is called a piecewise smooth function.

2.2.4. Assume that $f(x)$ is an absolutely integrable periodic function, the Fourier series expansion corresponding to $f(x)$ is

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\omega x+b_n\sin n\omega x), \quad (1)$$

where $\omega=\frac{2\pi}{T}$, and $a_n=\frac{2}{T}\int_I f(x)\cos n\omega x dx$,

$b_n=\frac{2}{T}\int_I f(x)\sin n\omega x dx$ for each non-negative integer n.

2.3. Fourier convergence theorem:

Suppose that $f(x)$ is an absolutely integrable periodic function, then for every continuous point x of $f(x)$, the Fourier series expansion of $f(x)$ is pointwise convergent to $f(x)$; the Fourier series expansion of $f(x)$ is pointwise convergent to $\frac{1}{2}[f(x^+)+f(x^-)]$, for the discontinuous points x of $f(x)$.

3. Problems and Discussions

In the following, we will propose two examples to demonstrate how to use the Fourier series method to approach (pointwise convergence) an absolutely integrable periodic function, and how to use Maple to help us solve problems. First, we find the Fourier series expansion of square wave function:

Problem 3.1. Find the Fourier series expansion of

$$f(x)=\begin{cases}1 & 0<x\le\pi\\ -1 & \pi<x<2\pi\end{cases} \quad (2)$$

on the open interval $(0,2\pi)$.

Solution Let the Fourier series expansion of $f(x)$ on the open interval $(0,2\pi)$ be

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

Since $f(x)$ is an odd function on $(0,2\pi)$, it follows that

$$a_0=\frac{1}{\pi}\cdot\int_0^{2\pi}f(x)dx=0,$$

and

$$a_n=\frac{1}{\pi}\cdot\int_0^{2\pi}f(x)\cos nx dx=0,$$

for each positive integer n.

On the other hand,

$$b_n=\frac{1}{\pi}\cdot\int_0^{2\pi}f(x)\sin nx dx$$

$$=\frac{1}{\pi}\cdot[\int_0^{\pi}\sin nx dx+\int_{\pi}^{2\pi}(-\sin nx)dx]$$

$$=\frac{1}{\pi}\cdot(-\frac{1}{n}\cos nx\Big|_0^{\pi}+\frac{1}{n}\cos nx\Big|_{\pi}^{2\pi})$$

$$=\frac{1}{\pi}\cdot\{[-(-1)^n\frac{1}{n}+\frac{1}{n}]+[\frac{1}{n}-(-1)^n\frac{1}{n}]\}$$

$$=\frac{2}{n\pi}[1-(-1)^n]$$

$$=\begin{cases}\frac{4}{n\pi} & if \quad n \quad is \quad odd\\ 0 & if \quad n \quad is \quad even\end{cases}. \quad (3)$$

Thus, the Fourier series expansion of $f(x)$ on the open interval $(0,2\pi)$ is

$$\frac{4}{\pi}\cdot\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin(2k-1)x. \quad (4)$$

Since $\lim_{x\to\pi^-}f(x)=1$ and $\lim_{x\to\pi^+}f(x)=-1$, it follows from Fourier convergence theorem that

$$\frac{4}{\pi}\cdot\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin(2k-1)x=\begin{cases}1 & 0<x<\pi\\ -1 & \pi<x<2\pi\\ 0 & x=\pi\end{cases}. \quad (5)$$

In addition, let $x=\frac{\pi}{2}$, then

$$1=\frac{4}{\pi}\cdot\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin(2k-1)\frac{\pi}{2}. \quad (6)$$

Therefore, we obtain the famous formula

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{2k-1}=\frac{\pi}{4}. \quad (7)$$

That is,

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}. \quad (8)$$

Next, we use Maple to verify the correctness of Eq. (8).

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>sum((-1)^(k+1)/(2*k-1),k=1..infinity);
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$$\frac{1}{4}\pi$$

Moreover, we use Maple to draw the graph of $f(x)$ on the interval $(0,2\pi)$:

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>f:=piecewise(0<x<=Pi,1,Pi<x<2*Pi,-1);
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$$f := \begin{cases} 1 & 0 < x \text{ and } x \le \pi \\ -1 & \pi < x \text{ and } x < 2\pi \end{cases}$$

>plot(f,x=0..2*Pi,discont=true);

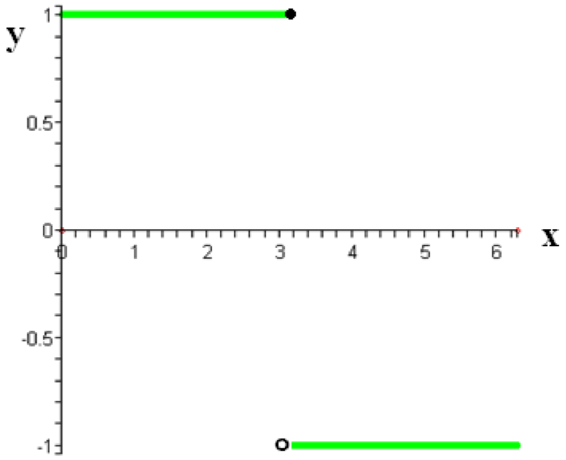

Figure 1. The graph of function $f(x)$ on $(0,2\pi)$.

On the other hand, we can determine b_n by using Maple:

>bn:=1/Pi*int(f*sin(n*x),x=0..2*Pi);

$$bn := \frac{-\frac{\cos(n\pi)-1}{n} + \frac{-\cos(n\pi)+2\cos(n\pi)^2-1}{n}}{\pi}$$

Since $\cos n\pi = (-1)^n$, it follows that the above answer is the same as Eq. (3). In addition, we also can list the first 12 terms of b_n:

>seq(bn,n=1..12);

$$\frac{4}{\pi}, 0, \frac{4}{3\pi}, 0, \frac{4}{5\pi}, 0, \frac{4}{7\pi}, 0, \frac{4}{9\pi}, 0, \frac{4}{11\pi}, 0$$

Next, we use s_1, s_4, s_{12} to approach $f(x)$, where

$$s_1 = \frac{4}{\pi}\sin x, \ s_4 = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x,$$

and

$$s_{12} = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x + \frac{4}{9\pi}\sin 9x + \frac{4}{11\pi}\sin 11x.$$

>s1:=sum(bn*sin(n*x),n=1..1);

$$s1 := \frac{4\sin(x)}{\pi}$$

>s4:=sum(bn*sin(n*x),n=1..5);

$$s4 := \frac{4\sin(x)}{\pi} + \frac{4}{3}\frac{\sin(3x)}{\pi} + \frac{4}{5}\frac{\sin(5x)}{\pi}$$

>s12:=sum(bn*sin(n*x),n=1..12);

$$s12 := \frac{4\sin(x)}{\pi} + \frac{4}{3}\frac{\sin(3x)}{\pi} + \frac{4}{5}\frac{\sin(5x)}{\pi} + \frac{4}{7}\frac{\sin(7x)}{\pi} + \frac{4}{9}\frac{\sin(9x)}{\pi} + \frac{4}{11}\frac{\sin(11x)}{\pi}$$

>plot({f,s1,s4,s12},x=0..2*Pi);

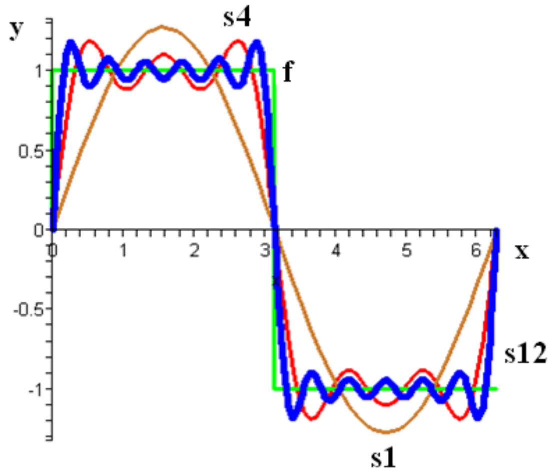

Figure 2. The brown, red, blue, and green curves represent the graphs of functions s_1, s_4, s_{12}, and $f(x)$ respectively; we can see that s_1, s_4, and s_{12} are gradually approaching $f(x)$.

Next, we study the Fourier series expansion of the triangular wave function:

Problem 3.2. Find the Fourier series expansion of

$$g(x) = \begin{cases} x+1 & -1 < x < 0 \\ -x+1 & 0 \le x < 1 \end{cases} \quad (9)$$

on the open interval $(-1,1)$.

Solution Assume that the Fourier series expansion of $g(x)$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x). \quad (10)$$

Since $g(x)$ is an even function, it follows that $b_n = 0$ for all positive integers n.

On the other hand,

$$a_0 = \int_{-1}^{1} f(x)dx = \int_{-1}^{0}(x+1)dx + \int_{0}^{1}(-x+1)dx = 1. \quad (11)$$

And for every positive integer n,

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x dx$$

$$=\int_{-1}^{0}(x+1)\cos n\pi x dx+\int_{0}^{1}(-x+1)\cos n\pi x dx$$

$$=\left(\frac{1}{n\pi}x\sin n\pi x+\frac{1}{n^2\pi^2}\cos n\pi x+\frac{1}{n\pi}\sin n\pi x\right)\Big|_{-1}^{0}$$

$$+\left(-\frac{1}{n\pi}x\sin n\pi x-\frac{1}{n^2\pi^2}\cos n\pi x+\frac{1}{n\pi}\sin n\pi x\right)\Big|_{0}^{1}$$

$$=\frac{2}{n^2\pi^2}[1-(-1)^n]$$

$$=\begin{cases}\dfrac{4}{n^2\pi^2} & if \quad n \quad is \quad odd\\ 0 & if \quad n \quad is \quad even\end{cases}. \tag{12}$$

Therefore, the Fourier expansion of $g(x)$ on $(-1,1)$ is

$$\frac{1}{2}+\sum_{k=1}^{\infty}\frac{4}{(2k-1)^2\pi^2}\cos(2k-1)\pi x. \tag{13}$$

Since $g(x)$ is continuous on open interval $(-1,1)$, using Fourier convergence theorem yields

$$\frac{1}{2}+\sum_{k=1}^{\infty}\frac{4}{(2k-1)^2\pi^2}\cos(2k-1)\pi x$$

$$=\begin{cases}x+1 & -1<x<0\\ -x+1 & 0\le x<1\end{cases}. \tag{14}$$

Moreover, if $x=0$ in Eq. (14), then

$$\frac{1}{2}+\sum_{k=1}^{\infty}\frac{4}{(2k-1)^2\pi^2}=1. \tag{15}$$

Hence, we obtain the famous formula

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}. \tag{16}$$

That is,

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}. \tag{17}$$

We employ Maple to verify the correctness of Eq. (17) as follows:

>sum(1/(2*k-1)^2,k=1..infinity);

$$\frac{1}{8}\pi^2$$

We also use Maple to draw the graph of $g(x)$ on $-1<x<1$:

>g:=piecewise(-1<x<0,x+1,0<=x<1,-x+1);

$$g:=\begin{cases}x+1 & -1<x \textbf{ and } x<0\\ -x+1 & 0\le x \textbf{ and } x<1\end{cases}$$

>plot(g,x=-1..1);

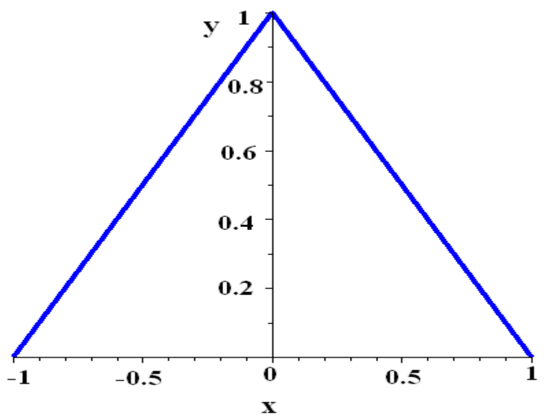

Figure 3. The graph of $g(x)$ on $-1<x<1$.

On the other hand, we can also calculate a_n by Maple:

>an:=int(g*cos(n*Pi*x),x=-1..1);

$$an:=-\frac{2(\cos(n\pi)-1)}{n^2\pi^2}$$

Thus, we can list the first 14 terms of a_n:

>seq(an,n=1..14);

$$\frac{4}{\pi^2},0,\frac{4}{9\pi^2},0,\frac{4}{25\pi^2},0,\frac{4}{49\pi^2},0,\frac{4}{81\pi^2},$$
$$0,\frac{4}{121\pi^2},0,\frac{4}{169\pi^2},0$$

In the following, we can use $t_1=\frac{4}{\pi}\sin x$ and $t_3=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin 3x+\frac{4}{5\pi}\sin 5x$ to approach $g(x)$:

>t1:=1/2+sum(an*cos(n*Pi*x),n=1..1);

$$t1:=\frac{1}{2}+\frac{4\cos(\pi x)}{\pi^2}$$

>t3:=1/2+sum(an*cos(n*Pi*x),n=1..3);

$$t3:=\frac{1}{2}+\frac{4\cos(\pi x)}{\pi^2}+\frac{4}{9}\frac{\cos(3\pi x)}{\pi^2}$$

>plot({g,t1,t3},x=-1..1);

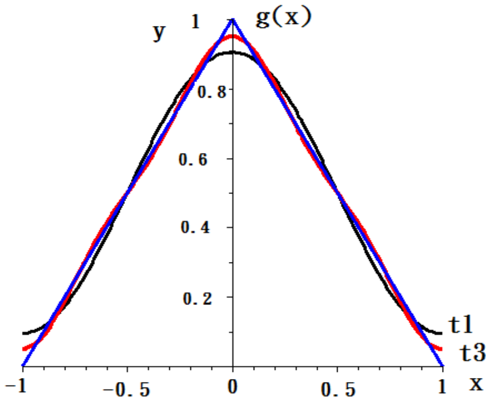

Figure 4. The black, red, and blue curves represent the graphs of functions t_1, t_3, and $g(x)$ respectively; we can see that t_1 and t_3 are gradually approaching $g(x)$.

4. Conclusion

It can be seen from the above discussions that we mainly use Fourier convergence theorem to find the Fourier series expansion of an absolutely integrable periodic function. And we know that Maple plays an important role in auxiliary problem solving. Even we can propose some functions, then find their Fourier series expansions and drawing their graphs by using Maple. In the future, we will use Fourier series to solve Laplace equation. These studies will serve as applications of Maple in mathematics education.

References

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